ON THE ROOTS OF THE LEGENDRE, LAGUERRE, AND HERMITE POLYNOMIALS

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ABSTRACT. *For several orthogonal polynomials, Cohen proved that their roots are the eigenvalues of symmetric tridiagonal matrices. In this paper, we give examples of this Cohen's result for the Legendre, Laguerre, and Hermite polynomials, which are useful in applications to quantum mechanics and numerical analysis.*

KEYWORDS: Laguerre and Hermite polynomials, Leverrier-Takeno's technique, Legendre polynomials.

INTRODUCTION

Here we consider the Legendre polynomials $P_n(x)$ [1, 2]:

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x), \dots \tag{1}$$

which verify the differential equation $(1 - x^2)y'' - 2xy' + l(l+1) = 0,\ \ l = 0, 1, 2, \dots$; Cohen [3, 4] showed that roots of $P_n(x) = 0$ are the proper values of the following symmetric tridiagonal matrix:

$$\boldsymbol{P}_n = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \square & \square & \square & \square & \square \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{15}} & \square & \square & \square & \square \\ \square & \frac{2}{\sqrt{15}} & 0 & \frac{3}{\sqrt{35}} & \square & \square & \square \\ \square & \square & \frac{3}{\sqrt{35}} & 0 & \ddots & \square & \square \\ \square & \square & \square & \ddots & \ddots & \ddots & \square \\ \square & \square & \square & \square & \ddots & 0 & g(n) \\ \square & \square & \square & \square & \square & g(n) & 0 \end{pmatrix}, \qquad g(n) = \frac{n-1}{\sqrt{(2n-1)(2n-3)}}. \tag{2}$$

Since the eigenvalues of a symmetric matrix are all real, it follows that the roots of the Legendre polynomials must all be real [1, 5, 6]. Moreover, the absence of nonzero terms along the leading diagonal of the matrix $\boldsymbol{P}_n$ implies that the eigenvalues are symmetrically distributed about the origin. Let's remember that the roots of $P_n(x)$ are important in the Gaussian quadrature [1, 7].

Similarly, the roots of the Laguerre polynomials $L_n(x)$ [8-10] as:

$$L_0(x)=1,\quad L_1(x)=1-x,\quad L_2(x)=\frac{1}{2}(x^2-4x+2),\quad L_3(x)=\frac{1}{6}(-x^3+9x^2-18x+6)\ ,\dots \tag{3}$$

which satisfy the differential equation $xy''+(1-x)y'+ny=0,\ n=0,1,2,\dots,$ are the eigenvalues of the symmetric matrix [3, 4]:

$$\boldsymbol{L}_n=\begin{pmatrix} 1 & 1 & \square & & \square & \square & \square & \square \\ 1 & 3 & 2 & & \square & \square & \square & \square \\ \square & 2 & 5 & & \ddots & \square & \square & \square \\ \square & \square & \ddots & & \ddots & \ddots & \square & \square \\ \square & \square & \square & \ddots & \ddots & & \ddots & \square \\ \square & \square & \square & \square & n-2 & 2n-3 & & n-1 \\ \square & \square & \square & \square & \square & n-1 & & 2n-1 \end{pmatrix}, \tag{4}$$

Hence, all solutions of $L_n(x)=0$ are real [11].

Besides, the Hermite polynomials $H_n(x)$ [12, 13] which are:

$$H_0(x)=1,\qquad H_1(x)=2x,\qquad H_2(x)=4x^2-2,\qquad H_3(x)=8x^3-12x,\ \dots \tag{5}$$

obey the differential equation $y''-2xy'+2ny=0,\ n=0,1,2,\dots$, have real roots [14-16] corresponding to the proper values of [17]:

$$\boldsymbol{H}_n=\begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & \square & \square & \square & \square & \square \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{2}{2}} & \square & \square & \square & \square \\ \square & \sqrt{\frac{2}{2}} & 0 & \sqrt{\frac{3}{2}} & & & \square \\ \square & \square & \ddots & \ddots & \ddots & \square & \square \\ \square & \square & \square & \ddots & \ddots & 0 & \sqrt{\frac{n-1}{2}} \\ \square & \square & \square & \square & \square & \sqrt{\frac{n-1}{2}} & 0 \end{pmatrix}, \tag{6}$$

which are symmetrically distributed about the origin.

In Sec. 2 we employ the Leverrier-Takeno's technique [18] to realize applications of (2, 4, 6).

SOME APPLICATIONS OF THE COHEN'S RESULTS

The characteristic equation of a matrix $\boldsymbol{A}_{n\,x\,n}$:

$$\lambda^n + a_1\,\lambda^{n-1} + a_2\,\lambda^{n-2} + \cdots + a_n = 0, \tag{7}$$

can be constructed via the Leverrier-Takeno's procedure:

$$a_1 = -s_1\,, \qquad a_2 = \tfrac{1}{2}[(s_1)^2 - s_2], \qquad a_3 = \tfrac{1}{6}[-(s_1)^3 + 3s_1\,s_2 - 2s_3],\ \ldots \tag{8}$$

where s_r is the trace of $\boldsymbol{A}^r$. Then we consider (2):

$$\boldsymbol{P}_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix}, \qquad \boldsymbol{P}_2^2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \qquad s_1 = a_1 = 0, \quad s_2 = \frac{2}{3}, \quad a_2 = -\frac{1}{3}\,,$$

thus (7) implies the equation $3\lambda^2 - 1 = 0$ in agreement with $P_2(x) = 0$. Similarly:

$$\boldsymbol{P}_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{15}} \\ 0 & \frac{2}{\sqrt{15}} & 0 \end{pmatrix}, \qquad \boldsymbol{P}_3^2 = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3\sqrt{5}} \\ 0 & \frac{3}{5} & 0 \\ \frac{2}{3\sqrt{5}} & 0 & \frac{4}{15} \end{pmatrix}, \qquad \boldsymbol{P}_3^3 = \begin{pmatrix} 0 & \frac{\sqrt{3}}{5} & 0 \\ \frac{\sqrt{3}}{5} & 0 & \frac{6}{5\sqrt{15}} \\ 0 & \frac{6}{5\sqrt{5}} & 0 \end{pmatrix},$$

Hence, $s_1 = a_1 = 0$, $s_2 = \frac{6}{5}$, $a_2 = -\frac{3}{5}$, $s_3 = a_3 = 0$, and from (7) we obtain $5\lambda^3 - 3\lambda = 0$ in harmony with $P_3(x) = 0$. Let's remember that the roots of Legendre polynomials are important in the Gaussian quadrature [7], in the study of electromagnetic radiation and the angular function for the hydrogen atom.

Besides, from (4):

$$\boldsymbol{L}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \qquad \boldsymbol{L}_2^2 = \begin{pmatrix} 2 & 4 \\ 4 & 10 \end{pmatrix}, \qquad s_1 = 4, \quad s_2 = 12, \quad a_1 = -4, \quad a_2 = 2,$$

then (7) gives $\lambda^2 - 4\lambda + 2 = 0$, equivalent to $L_2(x) = 0$; and:

$$\boldsymbol{L}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 5 \end{pmatrix}, \qquad \boldsymbol{L}_3^2 = \begin{pmatrix} 2 & 4 & 2 \\ 4 & 14 & 16 \\ 2 & 6 & 29 \end{pmatrix}, \qquad \boldsymbol{L}_3^3 = \begin{pmatrix} 6 & 18 & 18 \\ 18 & 78 & 108 \\ 18 & 108 & 177 \end{pmatrix},$$

therefore $s_1 = 9,\ s_2 = 45,\ s_3 = 261,\ a_1 = -9,\ a_2 = 18,\ a_3 = -6,$ and from (7) we deduce that $\lambda^3 - 9\lambda^2 + 18\lambda - 6 = 0$ in according with (3). The Laguerre polynomials participate in the radial function of hydrogen-like atoms [19] and diatomic molecules [20].

For the Hermite polynomials, we have:

$$\boldsymbol{H}_2 = \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & 0 \end{pmatrix}, \quad \boldsymbol{H}_2^2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad s_1 = a_1 = 0,\ s_2 = 1,\ a_2 = -\frac{1}{2},$$

implying the characteristic equation $2\lambda^2 - 1 = 0$, equivalent to $H_2(x) = 0$; and:

$$\boldsymbol{H}_3 = \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & 0 \\ \sqrt{\frac{1}{2}} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{H}_3^2 = \begin{pmatrix} \frac{1}{2} & 0 & \sqrt{\frac{1}{2}} \\ 0 & \frac{3}{2} & 0 \\ \sqrt{\frac{1}{2}} & 0 & 1 \end{pmatrix}, \quad \boldsymbol{H}_3^3 = \begin{pmatrix} 0 & \frac{3}{2}\sqrt{\frac{1}{2}} & 0 \\ \frac{3}{2}\sqrt{\frac{1}{2}} & 0 & \frac{3}{2} \\ 0 & \frac{3}{2} & 0 \end{pmatrix},$$

$s_1 = a_1 = 0,\ s_2 = 3,\ a_2 = -\frac{3}{2},\ s_3 = a_3 = 0,$ thus (7) gives the expression $2\lambda^3 - 3\lambda = 0$ which is compatible with $H_3(x) = 0.$ The Hermite polynomials are fundamental in the analysis of the harmonic oscillator in quantum physics.

Thus we have that the Leverrier-Takeno's process [18] allows to see that the eigenvalues of the matrices (2, 4, 6) are the roots of the Legendre, Laguerre [21, 22], and Hermite polynomials, respectively. Let's remember that the QR algorithm [23-26] is an efficient method to determine the proper values of a matrix.

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