ON THE GENERAL SOLUTION OF 2^{TH} ORDER LINEAR DIFFERENTIAL EQUATION

J. López-Bonilla, G. Posadas-Durán, O. Salas-Torres

ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México.
Email: lopezb@ipn.mx

ABSTRACT. *We employ a method of factorization to obtain the general solution of the second order linear differential equation, which is an alternative procedure to the usual Variation of Parameters method of Lagrange. We consider that our approach can be adapted to linear differential equations of the third and fourth order.*

KEYWORDS. Linear differential equation of second order, Variation of parameters, Factorization method.

INTRODUCTION

We consider the linear differential equation:

$$p(x)y'' + q(x)y' + r(x)y = \phi(x), \tag{1}$$

if the solution $y_1(x)$ of the corresponding homogeneous equation is known, then:

$$p y_1'' + q y_1' + r y_1 = 0. \tag{2}$$

In Sec. 2 we show that the factorization:

$$y(x) = y_1(x)\, v(x), \tag{3}$$

, allows to determine the general solution of (1), in harmony with the method of variation of parameters of Lagrange [1-3]. Our procedure is an alternative to several approaches to solve (1) [4-10], and we consider that it can be applied to differential equations of the third and fourth order [11, 12].

GENERAL SOLUTION VIA FACTORIZATION

If we employ (2) and (3) into (1) we obtain the expression:

$$p\, y_1\, v'' + (2p\, y_1' + q\, y_1)\, v' = \phi, \tag{4}$$

where we can introduce the function $u(x) = v'$ to deduce the equation:

$$u' + \left(\frac{q}{p} + \frac{2y_1'}{y_1}\right) u = \frac{\phi}{p\, y_1}\,, \tag{5}$$

that is:

$$\frac{d}{dx}\left(\frac{y_1^2}{W}\, u\right) = \frac{y_1\, \phi}{p\, W}\,, \qquad W = \exp\left(-\int^x \frac{q}{p}\, d\xi\right), \tag{6}$$

whose solution is immediate:

$$u = \frac{dv}{dx} = \frac{W}{y_1^2} \int^x \frac{y_1\, \phi}{p\, W}\, d\eta + c_2\, \frac{W}{y_1^2}\,. \tag{7}$$

Therefore, (3) and the integration of (7) imply the general solution of (1):

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x), \tag{8}$$

where:

$$y_2(x) = y_1(x) \int^x \frac{W}{y_1^2}\, d\eta\,, \tag{9}$$

$$y_p(x) = y_1(x) \int^x \frac{W(\varphi)}{y_1^2(\varphi)}\, d\varphi \int^\varphi \frac{y_1(\eta)\, \phi(\eta)}{p(\eta)\, W(\eta)}\, d\eta\,. \tag{10}$$

In (10) we can apply the method of integration by parts, thus giving:

$$y_p(x) \overset{(9)}{=} y_1(x) \left[\frac{y_2(x)}{y_1(x)} \int^x \frac{y_1\, \phi}{p\, W} d\eta - \int^x \frac{y_2\, \phi}{p\, W} d\eta\right] = y_2(x) \int^x \frac{y_1\, \phi}{p\, W}\, d\eta - y_1(x) \int^x \frac{y_2\, \phi}{p\, W}\, d\eta\,, \tag{11}$$

which is in agreement with the method of variation of parameters of Lagrange [1-3].

Our approach shows that the factorization (3) and one solution of (2) allow deduce the general solution of (1), without the *ansatz* of Lagrange in his procedure of variation of parameters.

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