# COMPLEX INTUITIONISTIC FUZZY SUBRINGS

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**ABSTRACT.** *In this paper, we defined the complex intuitionistic fuzzy subring and introduced some new concepts like Intuitionistic* $\pi-$*fuzzy sets and homogeneous complex intuitionistic fuzzy sets. Then, we investigated some of characteristics of complex intuitionistic fuzzy subring. The relationship between complex intuitionistic fuzzy subring and intuitionistic fuzzy subring is also investigated. It is found that every complex intuitionistic fuzzy subring yields two intuitionistic fuzzy subring. Finaly, we defined the image and inverse image of complex intuitionistic fuzzy subring under ring homomorphism, and thus studied their elementary properties.*

**KEYWORDS.** *Intuitionistic* $\pi-$*fuzzy set, intuitionistic* $\pi-$*fuzzy subring, homogeneous complex intuitionistic fuzzy set, complex intuitionistic fuzzy subring.*

## INTRODUCTION

After the introduction of the concept of fuzzy set by Zadeh (1965), many researches were conducted on the generalization of the notion of fuzzy set. Atanassov (1986) introduced the concept of intuitionistic fuzzy set. Hur *et al*., (2003) investigated intuitionistic fuzzy subgroups and subrings in 2003. The concept of the complex fuzzy sets was introduced (Ramot *et al.*, 2002). The concept of a complex intuitionistic fuzzy set was introduced by Alkouri *et al.*, (2012). In three recent papers, Alsarahead and Ahmad (2017a; 2017b; 2017c) introduced the concepts of complex fuzzy subgroup, complex fuzzy subring and complex intuitionistic fuzzy subgroup.

In this paper, we defined the complex intuitionistic fuzzy subrings and introduced some new concepts like intuitionistic $\pi-$fuzzy subring. Then, we investigated some of characteristics of complex intuitionistic fuzzy subrings. Finaly, we defined the image and inverse image of complex intuitionistic fuzzy subrings under ring homomorphism, and then we studied their properties.

## PRELIMINARIES

**Definition 1.** Let $A=\{(x,\mu_A(x),\nu_A(x)):x\in R\}$ be an intuitionistic fuzzy set of a ring $R$. Then, $A$ is said to be an intuitionistic fuzzy subring of $R$ if for all $x,y\in R$ the followings hold:

1. $\mu_A(x-y)\geq min\{\mu_A(x),\mu_A(y)\}$.
2. $\mu_A(xy)\geq min\{\mu_A(x),\mu_A(y)\}$.
3. $\nu_A(x-y)\leq max\{\nu_A(x),\nu_A(y)\}$.
4. $\nu_A(xy)\leq max\{\nu_A(x),\nu_A(y)\}$, Atanassov (1986).

**Definition 2.** A complex intuitionistic fuzzy set $A$, defined on a universe of discourse $U$, is characterized by membership and non-membership functions $\mu_A(x)=r_A(x)e^{i\omega_A(x)}$ and $\nu_A(x)=\hat{r}_A(x)e^{i\hat{\omega}_A(x)}$, respectively, that assign any element $x\in U$ a complex-valued grade of both membership and non-membership in $A$. By definition,

$A=\{(x,\mu_A(x),\nu_A(x)):x\in U\}$ where $r_A(x)+\hat{r}_A(x)\leq 1$. Alkouri and Salleh (2012).

**Definition 3.** Let $A$ and $B$ be two complex intuitionistic fuzzy subsets of $U$, with membership functions $\mu_A(x)=r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x)=r_B(x)e^{i\omega_B(x)}$, respectively, while the non-membership functions are $\nu_A(x)=\hat{r}_A(x)e^{i\hat{\omega}_A(x)}$ and $\nu_B(x)=\hat{r}_B(x)e^{i\hat{\omega}_B(x)}$, respectively. Then $A\cap B$ is given by:

$A\cap B=\{(x,\mu_{A\cap B}(x),\nu_{A\cap B}(x)):x\in U\}$ where

$$\mu_{A\cap B}(x)=min\{r_A(x),r_B(x)\}e^{i\,min\{\omega_A(x),\omega_B(x)\}}$$

$$\nu_{A\cap B}(x)=max\{\hat{r}_A(x),\hat{r}_B(x)\}e^{i\,max\{\hat{\omega}_A(x),\hat{\omega}_B(x)\}}$$

. Alkouri and Salleh (2012).

**Definition 4.** Let $A=\{(x,\mu_A(x),\nu_A(x)):x\in U\}$ be a intuitionistic fuzzy set. Then the set $A_\pi=\{(x,\gamma_{A_\pi}(x),\rho_{A_\pi}(x)):x\in U\}$ is said to be intuitionistic $\pi-$fuzzy set where $\gamma_{A_\pi}(x)=2\pi\mu_A(x)$ and $\rho_{A_\pi}(x)=2\pi\nu_A(x)$.

Note that the condition $\gamma_{A_\pi}(x)+\rho_{A_\pi}(x)\leq 2\pi$ is already satisfied. Alsarahead and Ahmad (2017c).

**Definition 5.** Let $A_\pi=\{(x,\gamma_{A_\pi}(x),\rho_{A_\pi}(x)):x\in U\}$ be an intuitionistic $\pi-$fuzzy set of a ring $R$. Then $A_\pi$ is said to be an intuitionistic $\pi-$fuzzy subring of $R$ if for all $x,y\in R$ the following hold:

1. $\gamma_{A_\pi}(x-y) \geq min\{\gamma_{A_\pi}(x), \gamma_{A_\pi}(y)\}$.
2. $\gamma_{A_\pi}(xy) \geq min\{\gamma_{A_\pi}(x), \gamma_{A_\pi}(y)\}$.
3. $\rho_{A_\pi}(x-y) \leq max\{\rho_{A_\pi}(x), \rho_{A_\pi}(y)\}$.
4. $\rho_{A_\pi}(xy) \leq max\{\rho_{A_\pi}(x), \rho_{A_\pi}(y)\}$.

**Proposition 6.** An intuitionistic $\pi-$fuzzy set $A_\pi$ is an intuitionistic $\pi-$fuzzy subring if and only if $A$ is an intuitionistic fuzzy subring.
**Proof.** Clear.

**Definition 7.** Let $A$ and $B$ be two complex intuitionistic fuzzy subsets of $G$, with membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$, respectively. While the non-membership functions are $\nu_A(x) = \hat{r}_A(x)e^{i\hat{\omega}_A(x)}$ and $\nu_B(x) = \hat{r}_B(x)e^{i\hat{\omega}_B(x)}$ respectively. Then
1. A complex intuitionistic fuzzy subset $A$ is said to be a homogeneous complex intuitionistic fuzzy set if for all $x, y \in G$ the following hold:

1. $r_A(x) \leq r_A(y)$ if and only if $\omega_A(x) \leq \omega_A(y)$.
2. $\hat{r}_A(x) \leq \hat{r}_A(y)$ if and only if $\hat{\omega}_A(x) \leq \hat{\omega}_A(y)$.

2. A complex intuitionistic fuzzy subset $A$ is said to be homogeneous with $B$, if for All $x, y \in G$ the following hold:

1. $r_A(x) \leq r_B(y)$ if and only if $\omega_A(x) \leq \omega_B(y)$.
2. $\hat{r}_A(x) \leq \hat{r}_B(y)$ if and only if $\hat{\omega}_A(x) \leq \hat{\omega}_B(y)$. Alsarahead and Ahmad (2017c).

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**Definition 8.** Let $A = \{(x, \mu_A(x), \nu_A(x)) : x \in R\}$ be a homogeneous complex intuitionistic fuzzy set of a ring $R$. Then $A$ is said to be a complex intuitionistic fuzzy subring of $R$ if for all $x, y \in R$ the following hold:

1. $\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$.
2. $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$.
3. $\nu_A(x-y) \leq max\{\nu_A(x), \nu_A(y)\}$.
4. $\nu_A(xy) \leq max\{\nu_A(x), \nu_A(y)\}$.

**Theorem 9.** Let $R$ be a ring and $A=\{(x,\mu_A(x),\nu_A(x)):x\in R\}$ be a homogeneous complex intuitionistic fuzzy set with membership function $\mu_A(x)=r_A(x)e^{i\omega_A(x)}$ and non-membership function $\nu_A(x)=\hat{r}_A(x)e^{i\hat{\omega}_A(x)}$. Then $A$ is a complex intuitionistic fuzzy subring of $R$ if and only if:

1. The intuitionistic fuzzy set $\overline{A}=\{(x,r_A(x),\hat{r}_A(x)):x\in R, r_A(x),\hat{r}_A(x)\in[0,1]\}$ is an intuitionistic fuzzy subring.
2. The intuitionistic $\pi-$fuzzy set $\underline{A}=\{(x,\omega_A(x),\hat{\omega}_A(x)):x\in R,\omega_A(x),\hat{\omega}_A(x)\in[0,2\pi]\}$ is an intuitionistic $\pi-$fuzzy subring.

**Proof.** Let $A$ be a complex intuitionistic fuzzy subring and $x,y\in R$. Then we have

$$\begin{aligned}
r_A(x-y)e^{i\omega_A(x-y)} &= \mu_A(x-y)\\
&\geq min\{\mu_A(x),\mu_A(y)\}\\
&= min\left\{r_A(x)e^{i\omega_A(x)},r_A(y)e^{i\omega_A(y)}\right\}\\
&= min\{r_A(x),r_A(y)\}e^{i\,min\{\omega_A(x),\omega_A(y)\}}
\end{aligned}$$

(since $A$ is homogeneous).

So $r_A(x-y)\geq min\{r_A(x),r_A(y)\}$ and $\omega_A(x-y)\geq min\{\omega_A(x),\omega_A(y)\}$. Also, we have

$$\begin{aligned}
r_A(xy)e^{i\omega_A(xy)} &= \mu_A(xy)\\
&\geq min\{\mu_A(x),\mu_A(y)\}\\
&= min\left\{r_A(x)e^{i\omega_A(x)},r_A(y)e^{i\omega_A(y)}\right\}\\
&= min\{r_A(x),r_A(y)\}e^{i\,min\{\omega_A(x),\omega_A(y)\}}
\end{aligned}$$

(since $A$ is homogeneous).

which implies $r_A(xy)\geq min\{r_A(x),r_A(y)\}$ and $\omega_A(xy)\geq min\{\omega_A(x),\omega_A(y)\}$. On the other hand

$$\begin{aligned}
\hat{r}_A(x-y)e^{i\hat{\omega}_A(x-y)} &= \nu_A(x-y)\\
&\leq max\{\nu_A(x),\nu_A(y)\}\\
&= max\left\{\hat{r}_A(x)e^{i\hat{\omega}_A(x)},\hat{r}_A(y)e^{i\hat{\omega}_A(y)}\right\}\\
&= max\{\hat{r}_A(x),\hat{r}_A(y)\}e^{i\,max\{\hat{\omega}_A(x),\hat{\omega}_A(y)\}}
\end{aligned}$$

(since $A$ is homogeneous).

So $\hat{r}_A(x-y)\leq max\{\hat{r}_A(x),\hat{r}_A(y)\}$ and $\hat{\omega}_A(x-y)\leq max\{\hat{\omega}_A(x),\hat{\omega}_A(y)\}$. Also, we have

$$\begin{aligned}
\hat{r}_A(xy)e^{i\hat{\omega}_A(xy)} &= \nu_A(xy)\\
&\leq max\{\nu_A(x),\nu_A(y)\}\\
&= max\left\{\hat{r}_A(x)e^{i\hat{\omega}_A(x)},\hat{r}_A(y)e^{i\hat{\omega}_A(y)}\right\}\\
&= max\{\hat{r}_A(x),\hat{r}_A(y)\}e^{i\,max\{\hat{\omega}_A(x),\hat{\omega}_A(y)\}}
\end{aligned}$$

(since $A$ is homogeneous).

which implies $\hat{r}_A(xy) \le max\{\hat{r}_A(x), \hat{r}_A(y)\}$ and $\hat{\omega}_A(xy) \le max\{\hat{\omega}_A(x), \hat{\omega}_A(y)\}$.
So $\overline{A}$ is an intuitionistic fuzzy subring and $\underline{A}$ is an intuitionistic $\pi-$fuzzy subring.
Conversely, let $\overline{A}$ be an intuitionistic fuzzy subring and $\underline{A}$ be an intuitionistic $\pi-$fuzzy subring.
So we have

$$\begin{aligned} r_A(x-y) &\ge min\{r_A(x), r_A(y)\} & \omega_A(x-y) &\ge min\{\omega_A(x), \omega_A(y)\} \\ r_A(xy) &\ge min\{r_A(x), r_A(y)\} & \omega_A(xy) &\ge min\{\omega_A(x), \omega_A(y)\} \\ \hat{r}_A(x-y) &\le max\{\hat{r}_A(x), \hat{r}_A(y)\} & \hat{\omega}_A(x-y) &\le max\{\hat{\omega}_A(x), \hat{\omega}_A(y)\} \\ \hat{r}_A(xy) &\le max\{\hat{r}_A(x), \hat{r}_A(y)\} & \hat{\omega}_A(xy) &\le max\{\hat{\omega}_A(x), \hat{\omega}_A(y)\}. \end{aligned}$$

Now,

$$\begin{aligned} \mu_A(x-y) = r_A(x-y)e^{i\omega_A(x-y)} &\ge min\{r_A(x), r_A(y)\}e^{i\,min\{\omega_A(x), \omega_A(y)\}} \\ &= min\left\{r_A(x)e^{i\omega_A(x)}, r_A(y)e^{i\omega_A(y)}\right\} \text{ (homogeneity).} \\ &= min\{\mu_A(x), \mu_A(y)\}. \end{aligned}$$

Also, we have

$$\begin{aligned} \mu_A(xy) = r_A(xy)e^{i\omega_A(xy)} &\ge min\{r_A(x), r_A(y)\}e^{i\,min\{\omega_A(x), \omega_A(y)\}} \\ &= min\left\{r_A(x)e^{i\omega_A(x)}, r_A(y)e^{i\omega_A(y)}\right\} \text{ (homogeneity)} \\ &= min\{\mu_A(x), \mu_A(y)\}. \end{aligned}$$

On the other hand

$$\begin{aligned} \nu_A(x-y) = \hat{r}_A(x-y)e^{i\hat{\omega}_A(x-y)} &\le max\{\hat{r}_A(x), \hat{r}_A(y)\}e^{i\,max\{\hat{\omega}_A(x), \hat{\omega}_A(y)\}} \\ &= max\left\{\hat{r}_A(x)e^{i\hat{\omega}_A(x)}, \hat{r}_A(y)e^{i\hat{\omega}_A(y)}\right\} \\ &= max\{\nu_A(x), \nu_A(y)\}. \end{aligned}$$

Also, we have

$$\begin{aligned} \nu_A(xy) = \hat{r}_A(xy)e^{i\hat{\omega}_A(xy)} &\le max\{\hat{r}_A(x), \hat{r}_A(y)\}e^{i\,max\{\hat{\omega}_A(x), \hat{\omega}_A(y)\}} \\ &= max\left\{\hat{r}_A(x)e^{i\hat{\omega}_A(x)}, \hat{r}_A(y)e^{i\hat{\omega}_A(y)}\right\} \\ &= max\{\nu_A(x), \nu_A(y)\}. \end{aligned}$$

So $A$ is a complex intuitionistic fuzzy subring.

**Theorem 10.** Let $\{A_i : i \in I\}$ be a collection of complex intuitionistic fuzzy subrings of a ring $R$. Then $\cap_{i\in I} A_i$ is a complex intuitionistic fuzzy subring.

**Proof.** For all $i \in I$ we have $r_{A_i}(x)$ is an intuitionistic fuzzy subring and $\omega_{A_i}(x)$ is an intuitionistic $\pi-$fuzzy subring (Theorem 9). Now, let $x, y \in G$. Then

$$\mu_{\cap_{i\in I}A_i}(x-y) = r_{\cap_{i\in I}A_i}(x-y)e^{i\omega_{\cap_{i\in I}A_i}(x-y)}$$

$$= min_{i\in I}\left\{r_{A_i}(xy)\right\}e^{i\,min_{i\in I}\left\{\omega_{A_i}(xy)\right\}}$$

$$\geq min_{i\in I}\left\{min\left\{r_{A_i}(x), r_{A_i}(y)\right\}\right\}e^{i\,min_{i\in I}\left\{min\left\{\omega_{A_i}(x),\omega_{A_i}(y)\right\}\right\}}$$

$$= min\left\{min_{i\in I}\left\{r_{A_i}(x)\right\}, min_{i\in I}\left\{r_{A_i}(y)\right\}\right\}e^{i\,min\left\{min_{i\in I}\left\{\omega_{A_i}(x)\right\}, min_{i\in I}\left\{\omega_{A_i}(y)\right\}\right\}}$$

$$= min\left\{min_{i\in I}\left\{r_{A_i}(x)\right\}e^{i\,min_{i\in I}\left\{\omega_{A_i}(x)\right\}}, min_{i\in I}\left\{r_{A_i}(y)\right\}e^{i\,min_{i\in I}\left\{\omega_{A_i}(y)\right\}}\right\}$$

$$= min\left\{\mu_{\cap_{i\in I}A_i}(x), \mu_{\cap_{i\in I}A_i}(y)\right\}.$$

Also, we have

$$\mu_{\cap_{i\in I}A_i}(xy) = r_{\cap_{i\in I}A_i}(xy)e^{i\omega_{\cap_{i\in I}A_i}(xy)}$$

$$= min_{i\in I}\left\{r_{A_i}(xy)\right\}e^{i\,min_{i\in I}\left\{\omega_{A_i}(xy)\right\}}$$

$$\geq min_{i\in I}\left\{min\left\{r_{A_i}(x), r_{A_i}(y)\right\}\right\}e^{i\,min_{i\in I}\left\{min\left\{\omega_{A_i}(x),\omega_{A_i}(y)\right\}\right\}}$$

$$= min\left\{min_{i\in I}\left\{r_{A_i}(x)\right\}, min_{i\in I}\left\{r_{A_i}(y)\right\}\right\}e^{i\,min\left\{min_{i\in I}\left\{\omega_{A_i}(x)\right\}, min_{i\in I}\left\{\omega_{A_i}(y)\right\}\right\}}$$

$$= min\left\{min_{i\in I}\left\{r_{A_i}(x)\right\}e^{i\,min_{i\in I}\left\{\omega_{A_i}(x)\right\}}, min_{i\in I}\left\{r_{A_i}(y)\right\}e^{i\,min_{i\in I}\left\{\omega_{A_i}(y)\right\}}\right\}$$

$$= min\left\{\mu_{\cap_{i\in I}A_i}(x), \mu_{\cap_{i\in I}A_i}(y)\right\}.$$

On other hand

$$\nu_{\cap_{i\in I}A_i}(x-y) = \hat{r}_{\cap_{i\in I}A_i}(x-y)e^{i\hat{\omega}_{\cap_{i\in I}A_i}(x-y)}$$

$$= max_{i\in I}\left\{\hat{r}_{A_i}(x-y)\right\}e^{i\,max_{i\in I}\left\{\hat{\omega}_{A_i}(x-y)\right\}}$$

$$\leq max_{i\in I}\left\{max\left\{\hat{r}_{A_i}(x), r_{A_i}(y)\right\}\right\}e^{i\,max_{i\in I}\left\{max\left\{\hat{\omega}_{A_i}(x),\hat{\omega}_{A_i}(y)\right\}\right\}}$$

$$= max\left\{max_{i\in I}\left\{\hat{r}_{A_i}(x)\right\}, max_{i\in I}\left\{\hat{r}_{A_i}(y)\right\}\right\} e^{i\,max\left\{max_{i\in I}\left\{\hat{\omega}_{A_i}(x)\right\}, max_{i\in I}\left\{\hat{\omega}_{A_i}(y)\right\}\right\}}$$

$$= max\left\{max_{i\in I}\left\{\hat{r}_{A_i}(x)\right\}e^{i\,max_{i\in I}\left\{\hat{\omega}_{A_i}(x)\right\}},\ max_{i\in I}\left\{\hat{r}_{A_i}(y)\right\}e^{i\,max_{i\in I}\left\{\hat{\omega}_{A_i}(y)\right\}}\right\}$$

$$= max\left\{\nu_{\cap_{i\in I}A_i}(x), \nu_{\cap_{i\in I}A_i}(y)\right\}.$$

Also, we have

$$\nu_{\cap_{i\in I}A_i}(xy) = \hat{r}_{\cap_{i\in I}A_i}(xy)e^{i\hat{\omega}_{\cap_{i\in I}A_i}(xy)}$$

$$= max_{i\in I}\left\{\hat{r}_{A_i}(xy)\right\}e^{i\,max_{i\in I}\left\{\hat{\omega}_{A_i}(xy)\right\}}$$

$$\leq max_{i\in I}\left\{max\left\{\hat{r}_{A_i}(x), \hat{r}_{A_i}(y)\right\}\right\}e^{i\,max_{i\in I}\left\{max\left\{\hat{\omega}_{A_i}(x), \hat{\omega}_{A_i}(y)\right\}\right\}}$$

$$= max\left\{max_{i\in I}\left\{\hat{r}_{A_i}(x)\right\}, max_{i\in I}\left\{\hat{r}_{A_i}(y)\right\}\right\} e^{i\,max\left\{max_{i\in I}\left\{\hat{\omega}_{A_i}(x)\right\}, max_{i\in I}\left\{\hat{\omega}_{A_i}(y)\right\}\right\}}$$

$$= max\left\{max_{i\in I}\left\{\hat{r}_{A_i}(x)\right\}e^{i\,max_{i\in I}\left\{\hat{\omega}_{A_i}(x)\right\}},\ max_{i\in I}\left\{\hat{r}_{A_i}(y)\right\}e^{i\,max_{i\in I}\left\{\hat{\omega}_{A_i}(y)\right\}}\right\}$$

$$= max\left\{\nu_{\cap_{i\in I}A_i}(x), \nu_{\cap_{i\in I}A_i}(y)\right\}.$$

**Definition 11.** Let $A = \{(x, \mu_A(x), \nu_A(x)) : x \in U\}$ be a complex intuitionistic fuzzy set with membership function $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and non-membership function $\nu_A(x) = \hat{r}_A(x)e^{i\hat{\omega}_A(x)}$. For $\alpha, \hat{\alpha} \in [0,1]$ and $\beta, \hat{\beta} \in [0,2\pi]$, the set $A_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})} = \{x \in U : r_A(x) \geq \alpha, \omega_A(x) \geq \beta, \hat{r}_A(x) \leq \hat{\alpha}, \hat{\omega}_A(x) \leq \hat{\beta}\}$ is called a level subset of the complex intuitionistic fuzzy subset $A$. In particular if $\beta = \hat{\beta} = 0$, then we get the level subset $A_{\alpha}^{\hat{\alpha}} = \{x \in U : r_A(x) \geq \alpha, \hat{r}_A(x) \leq \hat{\alpha}\}$. If $\alpha = \hat{\alpha} = 0$, then we get the level subset $A_{\beta}^{\hat{\beta}} = \{x \in U : \omega_A(x) \geq \beta, \hat{\omega}_A(x) \leq \hat{\beta}\}$. Alsarahead and Ahmad (2017c).

**Theorem 12.** Let $A = \{(x, \mu_A(x), \nu_A(x)) : x \in R\}$ be a complex intuitionistic fuzzy subring of $R$ with membership function $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and non-membership function $\nu_A(x) = \hat{r}_A(x)e^{i\hat{\omega}_A(x)}$, if $r_A(e) \geq \alpha$, $\omega_A(e) \geq \beta$, $\hat{r}_A(e) \leq \hat{\alpha}$ and $\hat{\omega}_A(e) \leq \hat{\beta}$. Then the level subset $A_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})}$ is a subring of $R$.

**Proof.** $e \in A_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})}$, so $A_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})} \neq \phi$. Let $x, y \in A_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})}$. Then we have $r_A(x) \geq \alpha$, $\omega_A(x) \geq \beta$, $\hat{r}_A(x) \leq \hat{\alpha}$ and $\hat{\omega}_A(x) \leq \hat{\beta}$, also, $r_A(y) \geq \alpha$, $\omega_A(y) \geq \beta$, $\hat{r}_A(y) \leq \hat{\alpha}$ and $\hat{\omega}_A(y) \leq \hat{\beta}$.

Now,

$$\begin{aligned} r_A(x-y)e^{i\omega_A(x-y)} = \mu_A(x-y) &\geq min\{\mu_A(x), \mu_A(y)\} \\ &= min\left\{r_A(x)e^{i\omega_A(x)}, r_A(y)e^{i\omega_A(y)}\right\} \\ &= min\{r_A(x), r_A(y)\}e^{i\,min\{\omega_A(x),\omega_A(y)\}} \end{aligned}$$

This implies

$$\begin{aligned} r_A(x-y) &\geq min\{r_A(x), r_A(y)\} \\ &\geq min\{\alpha,\alpha\} \\ &= \alpha. \end{aligned}$$

And

$$\begin{aligned} \omega_A(x-y) &\geq min\{\omega_A(x), \omega_A(y)\} \\ &\geq min\{\beta,\beta\} \\ &= \beta. \end{aligned}$$

Also, we have

$$\begin{aligned} \hat{r}_A(x-y)e^{i\hat{\omega}_A(x-y)} = \nu_A(x-y) &\leq max\{\nu_A(x), \nu_A(y)\} \\ &= max\left\{\hat{r}_A(x)e^{i\hat{\omega}_A(x)}, \hat{r}_A(y)e^{i\hat{\omega}_A(y)}\right\} \\ &= max\{\hat{r}_A(x), \hat{r}_A(y)\}e^{i\,max\{\hat{\omega}_A(x),\hat{\omega}_A(y)\}} \end{aligned}$$

This implies

$$\begin{aligned} \hat{r}_A(x-y) &\leq max\{\hat{r}_A(x), \hat{r}_A(y)\} \\ &\leq max\{\hat{\alpha},\hat{\alpha}\} \\ &= \hat{\alpha}. \end{aligned}$$

And

$$\begin{aligned} \hat{\omega}_A(x-y) &\leq max\{\hat{\omega}_A(x), \hat{\omega}_A(y)\} \\ &\leq max\{\hat{\beta},\hat{\beta}\} \\ &= \hat{\beta}. \end{aligned}$$

So $x - y \in A_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})}$. On the other hand

$$\begin{aligned} r_A(xy)e^{i\omega_A(xy)} = \mu_A(xy) &\geq min\{\mu_A(x), \mu_A(y)\} \\ &= min\left\{r_A(x)e^{i\omega_A(x)}, r_A(y)e^{i\omega_A(y)}\right\} \\ &= min\{r_A(x), r_A(y)\}e^{i\,min\{\omega_A(x),\omega_A(y)\}} \end{aligned}$$

This implies

$$r_A(xy) \geq min\{r_A(x), r_A(y)\}$$
$$\geq min\{\alpha, \alpha\}$$
$$= \alpha.$$

And

$$\omega_A(xy) \geq min\{\omega_A(x), \omega_A(y)\}$$
$$\geq min\{\beta, \beta\}$$
$$= \beta.$$

Also, we have

$$\hat{r}_A(xy)e^{i\hat{\omega}_A(xy)} = \nu_A(xy) \leq max\{\nu_A(x), \nu_A(y)\}$$
$$= max\left\{\hat{r}_A(x)e^{i\hat{\omega}_A(x)}, \hat{r}_A(y)e^{i\hat{\omega}_A(y)}\right\}$$
$$= max\{\hat{r}_A(x), \hat{r}_A(y)\}e^{i\,max\{\hat{\omega}_A(x), \hat{\omega}_A(y)\}}$$

This implies

$$\hat{r}_A(xy) \leq max\{\hat{r}_A(x), \hat{r}_A(y)\}$$
$$\leq max\{\hat{\alpha}, \hat{\alpha}\}$$
$$= \hat{\alpha}.$$

And

$$\hat{\omega}_A(xy) \leq max\{\hat{\omega}_A(x), \hat{\omega}_A(y)\}$$
$$\leq max\{\hat{\beta}, \hat{\beta}\}$$
$$= \hat{\beta}.$$

Thus $xy \in A_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})}$, therefore $A_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})}$ is a subring of $R$.

## HOMOMORPHISM

**Theorem 13.** Let $f : R \to S$ be a ring epimorphism. Let $A$ be an intuitionistic fuzzy subring of $R$ and $B$ be an intuitionistic fuzzy subring of $S$. Then the inverse image of $B$ is an intuitionistic fuzzy subring of $R$ and the image of $A$ is an intuitionistic fuzzy subring of $S$. Banerjee and Basnet (2003).

We are going to generalize this result to the complex intuitionistic fuzzy subrings.

**Definition 14.** Let $f : R \to S$ be a homomorphism. Let $A = \{(x, \mu_A(x), \ \nu_A(x)) : x \in R\}$ and $B = \{(x, \mu_B(x), \nu_B(x)) : x \in S\}$ be complex intuitionistic fuzzy subrings.

Then $C = \{(y, f(\mu_A)(y), f(\nu_A)(y)) : y \in S\}$ is called image of $A$, where

$$f(\mu_A)(y) = \begin{cases} \vee\{\mu_A(x) : x \in R, f(x) = y\} & if \quad f^{-1}(y) \neq \phi \\ 0, & otherwise \end{cases}$$

$$f(\nu_A)(y) = \begin{cases} \wedge\{\nu_A(x) : x \in R, f(x) = y\} & if \quad f^{-1}(y) \neq \phi \\ 1, & otherwise \end{cases}$$

for all $y \in S$.

The set $D = \{(x, f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x)) : x \in R\}$ is called inverse image of $B$, where $f^{-1}(\mu_B)(x) = \mu_B(f(x))$ and $f^{-1}(\nu_B)(x) = \nu_B(f(x))$ for all $x \in R$.

**Lemma 15.** Let $f : R \to S$ be a ring homomorphism. Let $A$ be a complex intuitionistic fuzzy subring of $R$ and $B$ be a complex intuitionistic fuzzy subring of $S$, with membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$, respectively, while the non-membership functions are $\nu_A(x) = \hat{r}_A(x)e^{i\hat{\omega}_A(x)}$ and $\nu_B(x) = \hat{r}_B(x)e^{i\hat{\omega}_B(x)}$, respectively. Then

1. $f(\mu_A)(y) = f(r_A)(y)e^{if(\omega_A)(y)}$.
2. $f(\nu_A)(y) = f(\hat{r}_A)(y)e^{if(\hat{\omega}_A)(y)}$.
3. $f^{-1}(\mu_B)(x) = f^{-1}(r_B)(x)e^{if^{-1}(\omega_B)(x)}$.
4. $f^{-1}(\nu_B)(x) = f^{-1}(\hat{r}_B)(x)e^{if^{-1}(\hat{\omega}_B)(x)}$.

**Proof**.(1)

$$\begin{aligned} f(\mu_A)(y) &= max_{f(x)=y}\mu_A(x) \\ &= max_{f(x)=y} r_A(x)e^{i\omega_A(x)} \\ &= max_{f(x)=y} r_A(x)e^{i\,max_{f(x)=y}\omega_A(x)} \\ &\text{(since } A \text{ is homogeneous)} \\ &= f(r_A)(y)e^{if(\omega_A)(y)}. \end{aligned}$$

2)

$$\begin{aligned} f(\nu_A)(y) &= min_{f(x)=y}\nu_A(x) \\ &= min_{f(x)=y} \hat{r}_A(x)e^{i\hat{\omega}_A(x)} \\ &= min_{f(x)=y} \hat{r}_A(x)e^{i\,min_{f(x)=y}\hat{\omega}_A(x)} \\ &\text{(since } A \text{ is homogeneous)} \\ &= f(\hat{r}_A)(y)e^{if(\hat{\omega}_A)(y)}. \end{aligned}$$

(3)

$$\begin{aligned} f^{-1}(\mu_B)(x) &= \mu_B\big(f(x)\big) \\ &= r_B(f(x))e^{i\omega_B(f(x))} \\ &= f^{-1}(r_B)(x)e^{if^{-1}(\omega_B)(x)}. \end{aligned}$$

(4)

$$\begin{aligned} f^{-1}(\nu_B)(x) &= \nu_B\big(f(x)\big) \\ &= \hat{r}_B(f(x))e^{i\hat{\omega}_B(f(x))} \\ &= f^{-1}(\hat{r}_B)(x)e^{if^{-1}(\hat{\omega}_B)(x)}. \end{aligned}$$

**Theorem 16.** Let $f: R \to S$ be a ring epimorphism. Let $A$ be a complex intuitionistic fuzzy subring of $R$ with membership function $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and non-membership function $\nu_A(x) = \hat{r}_A(x)e^{i\hat{\omega}_A(x)}$. Then the image of $A$ is a complex intuitionistic fuzzy subring of $S$.

**Proof.** Since $A$ is a complex intuitionistic fuzzy subring, then by (Theorem 9) $\{(x, r_A(x), \hat{r}_A(x)) : x \in R\}$ is an intuitionistic fuzzy subring and $\{(x, \omega_A(x), \hat{\omega}_A(x)) : x \in R\}$ is an intuitionistic $\pi-$fuzzy subring. Thus by (Theorem 13) and (Proposition 6) the image of $\{(x, r_A(x), \hat{r}_A(x)) : x \in R\}$ and $\{(x, \omega_A(x), \hat{\omega}_A(x)) : x \in R\}$ are intuitionistic fuzzy subring and intuitionistic $\pi-$fuzzy subring, respectively, therefore for all $x, y \in S$ we have:

$$f(r_A)(x-y) \geq \min\{f(r_A)(x), f(r_A)(y)\},\ f(r_A)(xy) \geq \min\{f(r_A)(x), f(r_A)(y)\}$$
$$f(\hat{r}_A)(x-y) \leq \max\{f(\hat{r}_A)(x), f(\hat{r}_A)(y)\},\ f(\hat{r}_A)(xy) \leq \max\{f(\hat{r}_A)(x), f(\hat{r}_A)(y)\}$$
$$f(\omega_A)(x-y) \geq \min\{f(\omega_A)(x), f(\omega_A)(y)\},\ f(\omega_A)(xy) \geq \min\{f(\omega_A)(x), f(\omega_A)(y)\}$$
$$f(\hat{\omega}_A)(x-y) \leq \max\{f(\hat{\omega}_A)(x), f(\hat{\omega}_A)(y)\} \text{ and } f(\hat{\omega}_A)(xy) \leq \max\{f(\hat{\omega}_A)(x), f(\hat{\omega}_A)(y)\}$$

Now, by Lemma 15

$$\begin{aligned} f(\mu_A)(x-y) &= f(r_A)(x-y)e^{if(\omega_A)(x-y)} \\ &\geq min\{f(r_A)(x), f(r_A)(y)\}e^{imin\{f(\omega_A)(x), f(\omega_A)(y)\}} \\ &= min\{f(r_A)(x)e^{if(\omega_A)(x)}, f(r_A)(y)e^{if(\omega_A)(y)}\} \\ &= min\{f(\mu_A)(x), f(\mu_A)(y)\}. \end{aligned}$$

Also,

$$\begin{aligned} f(\mu_A)(xy) &= f(r_A)(xy)e^{if(\omega_A)(xy)} \\ &\geq min\{f(r_A)(x), f(r_A)(y)\}e^{imin\{f(\omega_A)(x), f(\omega_A)(y)\}} \\ &= min\{f(r_A)(x)e^{if(\omega_A)(x)}, f(r_A)(y)e^{if(\omega_A)(y)}\} \end{aligned}$$

$$= min\{f(\mu_A)(x), f(\mu_A)(y)\}.$$

On the other hand

$$\begin{aligned} f(\nu_A)(x-y) &= f(\hat{r}_A)(x-y)e^{if(\hat{\omega}_A)(x-y)} \\ &\leq max\{f(\hat{r}_A)(x), f(\hat{r}_A)(y)\}e^{imax\{f(\hat{\omega}_A)(x), f(\hat{\omega}_A)(y)\}} \\ &= max\{f(\hat{r}_A)(x)e^{if(\hat{\omega}_A)(x)}, f(\hat{r}_A)(y)e^{if(\hat{\omega}_A)(y)}\} \\ &= max\{f(\nu_A)(x), f(\nu_A)(y)\}. \end{aligned}$$

Also,

$$\begin{aligned} f(\nu_A)(xy) &= f(\hat{r}_A)(xy)e^{if(\hat{\omega}_A)(xy)} \\ &\leq max\{f(\hat{r}_A)(x), f(\hat{r}_A)(y)\}e^{imax\{f(\hat{\omega}_A)(x), f(\hat{\omega}_A)(y)\}} \\ &= max\{f(\hat{r}_A)(x)e^{if(\hat{\omega}_A)(x)}, f(\hat{r}_A)(y)e^{if(\hat{\omega}_A)(y)}\} \\ &= max\{f(\nu_A)(x), f(\nu_A)(y)\}. \end{aligned}$$

**Theorem 17.** Let $f: R \to S$ be a ring epimorphism. Let $B$ be a complex intuitionistic fuzzy subring of $S$, with membership function $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$ and non-membership function $\nu_B(x) = \hat{r}_B(x)e^{i\hat{\omega}_B(x)}$. Then the inverse image of $B$ is a complex intuitionistic fuzzy subring of $R$.

**Proof.** Since $B$ is a complex intuitionistic fuzzy subring, then by (Theorem 9) $\{(x, r_B(x), \hat{r}_B(x)) : x \in S\}$ is an intuitionistic fuzzy subring and $\{(x, \omega_B(x), \hat{\omega}_B(x)) : x \in S\}$ is an intuitionistic $\pi-$fuzzy subring. Thus by (Theorem 15) and (Proposition 6) the inverse image of $\{(x, r_B(x), \hat{r}_B(x)) : x \in S\}$ and $\{(x, \omega_B(x), \hat{\omega}_B(x)) : x \in S\}$ are intuitionistic fuzzy subring and intuitionistic $\pi-$fuzzy subring, respectively, therefore for all $x, y \in R$ we have:

$$f^{-1}(r_B)(x-y) \geq \min\{f^{-1}(r_B)(x), f^{-1}(r_B)(y)\},$$
$$f^{-1}(r_B)(xy) \geq \min\{f^{-1}(r_B)(x), f^{-1}(r_B)(y)\},$$
$$f^{-1}(\hat{r}_B)(x-y) \leq \max\{f^{-1}(\hat{r}_B)(x), f^{-1}(\hat{r}_B)(y)\},$$
$$f^{-1}(\hat{r}_B)(xy) \leq \max\{f^{-1}(\hat{r}_B)(x), f^{-1}(\hat{r}_B)(y)\},$$
$$f^{-1}(\omega_B)(x-y) \geq \min\{f^{-1}(\omega_B)(x), f^{-1}(\omega_B)(y)\},$$
$$f^{-1}(\omega_B)(xy) \geq \min\{f^{-1}(\omega_B)(x), f^{-1}(\omega_B)(y)\},$$
$$f^{-1}(\hat{\omega}_B)(x-y) \leq \max\{f^{-1}(\hat{\omega}_B)(x), f^{-1}(\hat{\omega}_B)(y)\} \text{ and}$$
$$f^{-1}(\hat{\omega}_B)(xy) \leq \max\{f^{-1}(\hat{\omega}_B)(x), f^{-1}(\hat{\omega}_B)(y)\}$$

Now, by Lemma 15

$$
\begin{aligned}
f^{-1}(\mu_B)(x-y) &= f^{-1}(r_B)(x-y)e^{if^{-1}(\omega_B)(x-y)} \\
&\geq min\{f^{-1}(r_B)(x), f^{-1}(r_B)(y)\}e^{imin\{f^{-1}(\omega_B)(x), f^{-1}(\omega_B)(y)\}} \\
&= min\left\{ f^{-1}(r_B)(x)e^{if^{-1}(\omega_B)(x)}, f^{-1}(r_B)(y)e^{if^{-1}(\omega_B)(y)} \right\} \\
&= min\{f^{-1}(\mu_B)(x), f^{-1}(\mu_B)(y)\}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
f^{-1}(\mu_B)(xy) &= f^{-1}(r_B)(xy)e^{if^{-1}(\omega_B)(xy)} \\
&\geq min\{f^{-1}(r_B)(x), f^{-1}(r_B)(y)\}e^{imin\{f^{-1}(\omega_B)(x), f^{-1}(\omega_B)(y)\}} \\
&= min\left\{ f^{-1}(r_B)(x)e^{if^{-1}(\omega_B)(x)}, f^{-1}(r_B)(y)e^{if^{-1}(\omega_B)(y)} \right\} \\
&= min\{f^{-1}(\mu_B)(x), f^{-1}(\mu_B)(y)\}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
f^{-1}(\nu_B)(x-y) &= f^{-1}(\hat{r}_B)(x-y)e^{if^{-1}(\hat{\omega}_B)(x-y)} \\
&\leq max\{f^{-1}(\hat{r}_B)(x), f^{-1}(\hat{r}_B)(y)\}e^{imax\{f^{-1}(\hat{\omega}_B)(x), f^{-1}(\hat{\omega}_B)(y)\}} \\
&= max\left\{ f^{-1}(\hat{r}_B)(x)e^{if^{-1}(\hat{\omega}_B)(x)}, f^{-1}(\hat{r}_B)(y)e^{if^{-1}(\hat{\omega}_B)(y)} \right\} \\
&= max\{f^{-1}(\nu_B)(x), f^{-1}(\nu_B)(y)\}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
f^{-1}(\nu_B)(xy) &= f^{-1}(\hat{r}_B)(xy)e^{if^{-1}(\hat{\omega}_B)(xy)} \\
&\leq max\{f^{-1}(\hat{r}_B)(x), f^{-1}(\hat{r}_B)(y)\}e^{imax\{f^{-1}(\hat{\omega}_B)(x), f^{-1}(\hat{\omega}_B)(y)\}} \\
&= max\left\{ f^{-1}(\hat{r}_B)(x)e^{if^{-1}(\hat{\omega}_B)(x)}, f^{-1}(\hat{r}_B)(y)e^{if^{-1}(\hat{\omega}_B)(y)} \right\} \\
&= max\{f^{-1}(\nu_B)(x), f^{-1}(\nu_B)(y)\}.
\end{aligned}
$$

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