SOR METHOD FOR THE IMPLICIT FINITE DIFFERENCE SOLUTION OF TIME-FRACTIONAL DIFFUSION EQUATIONS

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ABSTRACT. *In this study, we derive an unconditionally implicit finite difference approximation equation from the discretization of the one-dimensional linear time fractional diffusion equations by using the Caputo's time fractional derivative. Then this approximation equation hence will be used to generate the corresponding system of linear equations. The approximation solution of the linear system is described via the implementation of Successive Over-Relaxation (SOR) iterative method. An example of the problem is presented to illustrate the effectiveness of SOR method. The findings of this study show that the proposed iterative method is superior compared with the Gauss-Seidel iterative method.*

KEYWORDS. Caputo's fractional derivative, Implicit Finite Difference Scheme, SOR Method

INTRODUCTION

Presently, a lot of in modeling of diffusion processes is found in the natural world. Therefore, fractional partial differential equations (FPDEs) have attracted many researchers from various fields (Mainardi, 1997; Diethelm & Freed, 1999; Liu, *et al.*, 2004; Meerschaert, *et al.*, 2004) to study the numerical and/or analytical solutions of the problems. For instance, a fractional derivative replaces the first-order time partial derivative in a diffusion model and lead to slower diffusion (Mainardi, 1997). For a one-dimensional diffusion model with constant coefficients, analytical solutions are available using.

For the numerical solution of the fractional diffusion equations (FDE), many proposed methods have been initiated such as transform methods (Mainardi, 1997; Chaves, 1998; Agrawal, 2002), finite elements together with the method of lines (Liu, *et al.,* 2004, El-Kahlout, 2008), explicit and implicit finite difference methods (Liu, *et al.*, 2006; Meerschaert, *et al.*, 2004; Shen, *et al.*, 2005, Diego, 2008; Sweilam, *et al.*, 2012). Even though the explicit methods are conditionally stable, this finite difference schemes are available in the literature, (Yuste, *et al.*, 2005; Yuste, 2006).

In this paper, the main objective is to get numerical solutions of the one-dimensional time fractional parabolic partial differential equation (TPPDE's) iteratively in which the TPPDE's problem can be defined as

$$\frac{\partial^{\alpha} U(x,t)}{\partial t^{\alpha}} = a(x)\frac{\partial^{2} U(x,t)}{\partial x^{2}} + b(x)\frac{\partial U(x,t)}{\partial x} + c(x)U(x,t) \tag{1.1}$$

Before constructing the discrete equation of Eq. (1.1) in order to get its numerical solutions, the following are some basic definitions for fractional derivatives which are used in the paper.

Definition 1. (Zhang, 2009) The Riemann-Liouville fractional integral operator, J^{α} of order-α is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt\,,\ \alpha > 0\,,\ x > 0 \tag{1.2}$$

Definition 2. (Zhang, 2009) The Caputo's fractional partial derivative operator, D^{α} of order -α is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0 \tag{1.3}$$

with $m-1 < \alpha \le m, \mathrm{m} \in \mathrm{N},\ x > 0$

According to previous studies, many studies have been conducted to show the efficiency of the SOR method (Youssef, 2012; Sun, 2005; Starke, *et al.*, 1991; Hadjidimos, 2000). However, there is no SOR method in the literature for solving Time-Fractional Diffusion equation. Therefore this paper attempts to investigate the Full-Sweep Successive Over-Relaxation (FSSOR) iterative method which is compared with the Full-Sweep Gauss-Seidel (FSGS) iterative method for solving Problem (1.1) with variable coefficients. To prove the efficiency of this method, we use the usual Caputo's implicit finite difference approximations for the non-local fractional derivative operator, which is first order consistent and unconditionally stable for Problem (1.1) with Dirichlet boundary conditions. According to Problem (1.1), we restrict our attention to the finite space domain $0 \le x \le \gamma$, with $0 < \alpha < 1$ and the parameter α refers to the fractional order of time derivative. For simplicity, we also assume the initial and boundary conditions of Problem (1.1) given as

$$U(0,t) = g_0(t),\ U(\ell,t) = g_1(t),$$

and the initial condition

$$U(x,0) = f(x),.$$

where $g_0(t), g_1(t),$ and $f(x),$ are given functions. To discretize the the time fractional derivative in Eq. (1.1), we consider Caputo's fractional partial derivative of order α, defined by (Zhang, 2009; Young, 1954; Young, 1972),

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-1)} \int_0^{\infty} \frac{\partial u(x,s)}{\partial t} (t-s)^{-\alpha} ds, \quad t > 0, \quad 0 < \alpha < 1 \tag{1.3}$$

The organisation of the paper is as follows: In Section 2, an approximate formula of the fractional derivative and numerical procedure for solving time fractional diffusion equation (1.2) by means of the implicit finite difference method are given. In Section 3, formulation of the FSSOR iterative method will be discussed in Section 4 shows numerical experiment and conclusion is given in Section 5.

Caputo's Finite Difference Approximation

We introduce the basic ideas for the numerical solution of the time fractional diffusion equation (1.1) by implicit finite difference in this section. For some positive integers *m* and *n*, the grids sizes in space and time directions for the finite difference algorithm are defined as $h = \Delta x = \dfrac{\gamma - 0}{m}$ and $k = \Delta t = \dfrac{T}{n}$ respectively. The grids point in the space interval $[0,\gamma]$ are the numbers $x_i = ih,\ i = 0,1,2,\ldots,m$ and the grid points in the time interval $[0,T]$ are labeled

$t_j = jk,\ j = 0,1,2,\ldots,$.The values of the function $U(x,t)$ a at the grids point are denoted $U_{i,j} = U(x_i, t_j)$.

A discrete approximation to the fractional derivative (1.3) can be obtained by a simple quadrature formula as follows (Zhang, 2009):

$$\frac{\partial^\alpha U(x_i,t_n)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{\partial U(x_i,s)}{\partial t}(t_n - s)^{-\alpha} ds$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{n} \int_{(j-1)k}^{jk} \left[\frac{U_{i,j} - U_{i,j-1}}{k} + O(k) \right] (nk - s)^{-\alpha} ds$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} \sum_{j=1}^{n} \int_{(j-1)k}^{jk} \left\{ \left[\frac{U_{i,j} - U_{i,j-1}}{k} + O(k) \right] \left[(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha} \right] \right\} k^{1-\alpha}$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} \frac{1}{k^\alpha} \sum_{j=1}^{n} (U_{i,j} - U_{i,j-1}) \left[(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha} \right]$$

$$+ \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} \sum_{j=1}^{n} \left[(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha} \right] O(k^{2-\alpha}), \qquad (2.1)$$

Let us define

$$\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} \frac{1}{k^\alpha}$$

and

$$\omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha},$$

Then the discrete approximation of Eq.(2.1)

$$\frac{\partial^\alpha U(x_i,t_n)}{\partial t^\alpha} = \sigma_{\alpha,k} \sum_{j=0}^{n} \omega_j^{(\alpha)} (U_{i,n-j+1} - U_{i,n-j+1}) + \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} n^{1-\alpha} O(k^{2-\alpha})$$

$$= \sigma_{\alpha,k} \sum_{j=0}^{n} \omega_j^{(\alpha)} (U_{i,n-j+1} - U_{i,n-j}) + O(k) \qquad (2.2)$$

Hence, Eq.(2.2) can be indicated as

$$\frac{\partial^\alpha U(x_i,t_n)}{\partial t^\alpha} = D_t^{(\alpha)} U_{i,n} + O(k)$$

and the first order approximation method for the computation of Caputo's fractional partial derivative is then stated as the following expression

$$D_t^{(\alpha)} U_{i,n} = \sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} (U_{i,n-1+j} - U_{i,n-j}) \qquad (2.3)$$

Using Eq. (2.3) and the implicit finite difference discretization scheme, the discrete equation of Problem (1.1) to the grid point centered at $(x_i, t_j) = (ih, nk)$,is given as

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(U_{i,n-j+1}-U_{i,n-j})$$
$$=a_i\frac{1}{h^2}(U_{i-1,n}-2U_{i,n}+U_{i+1,n})+b_i\frac{1}{2h}(U_{i+1,n}-U_{i-1,n})+c_iU_{i,n}, \qquad (2.4)$$

for $i=1,2...,m-1$.

Thus, according to Eq. (2.4), the approximation equation is known as the fully implicit finite difference approximation equation which is consistent first order accuracy in time and second order in space. For simplicity, Eq.(2.4) for $n \ge 2$ can be rewritten as

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(U_{i,n-j+1}-U_{i,n-j})=(\frac{a_i}{h^2}-\frac{b_i}{2h})U_{i-1,n}+(c_i-\frac{2a_i}{h^2})U_{i,n}+(\frac{a_i}{h^2}+\frac{b_i}{2h})U_{i+1,n},$$

$$\therefore \sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(U_{i,n-j+1}-U_{i,n-j})=p_iU_{i-1,n}+q_iU_{i,n}+r_iU_{i+1,n}$$

where

$$p_i=\frac{a_i}{h^2}-\frac{b_i}{2h},\; q_i=c_i-\frac{2a_i}{h^2},\; r_i=\frac{a_i}{h^2}+\frac{b_i}{2h}.$$

Finally, we get for $n = 1$, $\omega_j^{(\alpha)}=1$

$$-p_iU_{i-1,1}+q_i^*U_{i,1}-r_iU_{i+1,1}=f_i^j, \quad i=1,2,\ldots,m-1 \qquad (2.5)$$

where $q_i^*=\sigma_{\alpha,k}-q_i$, and $f_i^j=\sigma_{\alpha,k}u_1^0$. Again Eq. (2.5) can be expressed in a matrix form as

$$A\underset{\sim}{U}=\underset{\sim}{f} \qquad (2.6)$$

(2.6)

where

$$A=\begin{bmatrix} q* & -r & & & & \\ -p & q* & -r & & & \\ & -p & q* & -r & & \\ & & \ddots & \ddots & \ddots & \\ & & & -p & q* & -r \\ & & & & -p & q* \end{bmatrix}_{(m-1)x(m-1)},$$

$$\underset{\sim}{U}=\begin{bmatrix}U_{11} & U_{21} & U_{31} & \cdots & U_{m-2,1} & U_{m-1,1}\end{bmatrix}^T,$$

$$\underset{\sim}{f}=\begin{bmatrix}U_{11}+p_1U_{01} & U_{21} & U_{31} & \cdots & U_{m-2,1} & U_{m-1,1}+p_{m-1}U_{m,1}\end{bmatrix}^T.$$

METHODOLOGY

Based on the tridiagonal linear system in Eq. (2.6), it is clear that the characteristic of its coefficient matrix has large scale and sparse. Actually, the concept of various iterative methods has been initiated and conducted by many researchers such as, Young (Young, 1954; Young, 1971; Young, 1972), Hackbusch (Hackbusch, 1995), Saad (Saad, 1996), Evans

(Evans, 1985), Yousif and Evans (Yousif, *et al.*, 1995), and Othman and Abdullah (Othman, *et al.*, 2000). To solve the tridiagonal linear system, Young (Young, 1954; Young, 1971; Young, 1972), initiated Successive Over-Relaxation (SOR) method, which is the most known and widely used iterative techniques to solve in solving any linear systems. Due to the advantages of FSSOR method, let the coefficient matrix A in (2.6) be expressed as summation of the three matrices

$$A = D - L - V \tag{3.1}$$

where D, L and V are diagonal, lower triangular and upper triangular matrices respectively.
Thus, SOR iterative method can be defined generally as

$$\tilde{U}^{(k+1)} = (D - \omega L)^{-1} [\,\omega V + (1-\omega)D\,]\tilde{U}^{(k)} + (D - \omega L)^{-1} f \tag{3.2}$$

where $\tilde{U}^{(k)}$ represents an unknown vector at kth iteration. The implementation of the SOR iterative method can be described in Algorithm 1.

Algorithm 1: SOR method

i. Initialize $\tilde{U} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$.
ii. For $i = 1,2,\ldots,n-1$ and $j = 1,2,\ldots,m-1$ assign
$$U_{i,j}^{(k+1)} = (1-\omega)U_{i+1,j}^{(k)} + \frac{\omega}{q_i^*}\left(p_i U_{i-1,1} + r_i U_{i+1,1} + f_{i,1}\right)$$
iii. Convergence test. If the convergence criterion i.e. $\left\| \tilde{U}^{(k+1)} - \tilde{U}^{(k)} \right\| \le \varepsilon = 10^{-10}$ is satisfied, go to Step (iv). Otherwise go back to Step (ii).
iv. Display approximate solutions.

RESULT AND DISCUSSION

In order to verify the effectiveness of the Full-Sweep Gauss-Seidel (FSGS) and Full-Sweep Successive Over-Relaxation (FSSOR) iterative methods, one example of the time fractional diffusion equation was tested. In comparison, three criteria will be considered for both iterative methods such as number of iterations (K), execution time (second) and maximum absolute error at two different values of $\alpha = 0.50$ and $\alpha = 0.75$. During the implementation of the point iterations, the convergence test considered the tolerance error, $\varepsilon = 10^{-10}$.

Examples 1:

Let us consider the following time fractional initial boundary value problem (Ali S *et al*, 2013)

$$\frac{\partial^{\alpha} U(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 U(x,t)}{\partial x^2}, \quad 0 < \alpha \le 1, 0 \le x \le \gamma, \quad t > 0, \tag{4.1}$$

where the boundary conditions are given in fractional terms

$$U(0,t) = \frac{2kt^{\alpha}}{\Gamma(\alpha+1)}, \; U(\ell,t) = \ell^2 + \frac{2kt^{\alpha}}{\Gamma(\alpha+1)}, \tag{4.2a}$$

and the initial condition

$$U(x,0) = x^2. \tag{4.2b}$$

From Problem (4.1), as taking $\alpha = 1$, it can be seen that Equation (4.1) can be reduced to the standard diffusion equation

$$\frac{\partial U(x,t)}{\partial t}=\frac{\partial^2 U(x,t)}{\partial x^2},\quad 0\le x\le\gamma,\quad t>0, \tag{4.3}$$

with the initial and boundary conditions

$$U(x,0)=x^2,\qquad U(0,t)=2kt,\qquad U(\ell,t)=\ell^2+2kt,$$

Then the analytical solution of Problem (4.3) is obtained as follows

$$U(x,t)=x^2+2kt.$$

Now by applying the series

$$U(x,t)=\sum_{n=0}^{m-1}\frac{\partial^n U(x,0)}{\partial t^n}\frac{t^n}{n!}+\sum_{n=1}^{\infty}\sum_{i=0}^{m-1}\frac{\partial^{mn+i}U(x,0)}{\partial t^{mn+i}}\frac{t^{n\alpha+i}}{\Gamma(n\alpha+i+1)}$$

to $U(x,t)$ for $0<\alpha\le 1$, it can be shown that the analytical solution of Problem (4.1) is given as

$$U(x,t)=x^2+2k\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Examples 2:

Let us consider the following time fractional initial boundary value problem (Ali S *et al*, 2013)

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha}=\frac{1}{2}x^2\frac{\partial^2 U(x,t)}{\partial x^2},\quad 0<\alpha\le 1, 0\le x\le\gamma,\quad t>0, \tag{4.4}$$

where the boundary conditions are given in fractional terms

$$U(0,t)=0,\quad U(1,t)=e^t,$$

(4.5a)

and the initial condition

$$U(x,0)=x^2.$$

(4.5b)

From Problem (4.4), as taking $\alpha=1$, it can be seen that Equation (4.4) can be reduced to the standard diffusion equation

$$\frac{\partial U(x,t)}{\partial t}=\frac{1}{2}x^2\frac{\partial^2 U(x,t)}{\partial x^2},\quad 0\le x\le\gamma,\quad t>0, \tag{4.6}$$

Then the analytical solution of Problem (4.6) is obtained as follows

$$U(x,t)=x^2e^t$$

Now by applying the series

$$U(x,t)=\sum_{n=0}^{m-1}\frac{\partial^n U(x,0)}{\partial t^n}\frac{t^n}{n!}+\sum_{n=1}^{\infty}\sum_{i=0}^{m-1}\frac{\partial^{mn+i}U(x,0)}{\partial t^{mn+i}}\frac{t^{n\alpha+i}}{\Gamma(n\alpha+i+1)}$$

to $U(x,t)$ for $0<\alpha\le 1$, it can be shown that the analytical solution of Problem (4.4) is given as

$$U(x,t)=x^2\left[1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right]$$

All results of numerical experiments for Problem (4.1) and Problem (4.2), obtained from implementation of FSGS and FSSOR iterative methods are recorded in Table 1 and Table 2 at different values of mesh sizes, M = 256, 512, 1024, 2048 and 4096.

Table 1. Comparison of Number Iterations (K), The Execution Time (Seconds) and Maximum Errors for the iterative methods using example at $\alpha = 0.50, 0.75$.

M	Method	α = 0.50			α = 0.75		
		K *(Number Iterations)*	**Time** *(Second)*	**Max Error**	**K** *(Number Iterations)*	**Time** *(Second)*	**Max Error**
256	**FSGS**	230579	1.46	7.6181e-6	282947	1.99	5.3418e-7
	FSSOR	1552	0.01	2.9290e-5	1764	0.02	2.8576e-6
512	**FSGS**	817596	10.24	3.3005e-6	1000946	13.98	2.1293e-6
	FSSOR	2984	0.03	7.2994e-6	3376	0.04	3.5758e-7
1024	**FSGS**	2853149	71.25	7.1831e-6	3482930	97.32	8.5184e-6
	FSSOR	5890	0.09	1.8275e-6	6615	0.13	4.4743e-8
2048	**FSGS**	9767783	487.01	2.7356e-5	11884877	664.92	3.4082e-5
	FSSOR	11258	0.33	4.6509e-7	12859	0.42	1.0835e-8
4096	**FSGS**	32773526	3266.51	1.0899e-4	39754285	48406.87	1.3610e-4
	FSSOR	21708	1.27	1.2600e-7	2427	1.56	2.9852e-8

Table 2. Comparison of Number Iterations (K), The Execution Time (Seconds) and Maximum Errors for the iterative methods using example at $\alpha = 0.50, 0.75$.

M	Method	α = 0.50			α = 0.75		
		K *(Number Iterations)*	**Time** *(Second)*	**Max Error**	**K** *(Number Iterations)*	**Time** *(Second)*	**Max Error**
256	**FSGS**	21017	37.73	9.97e-05	13601	5.92	9.86e-05
	FSSOR	7292	35.86	9.96e-05	4715	2.23	9.84e-05
512	**FSGS**	77231	343.63	1.00e-04	50095	42.17	9.90e-05
	FSSOR	26884	261.56	9.98e-05	17417	16.68	9.87e-05
1024	**FSGS**	281598	2747.34	1.02e-04	183181	339.85	1.01e-04
	FSSOR	98422	1916.28	1.00e-04	63298	123.01	9.96e-05
2048	**FSGS**	1017140	68285.36	1.09e-04	663971	2454.53	1.08e-05
	FSSOR	357258	14064.44	1.04e-04	232784	1007.47	1.03e-05
4096	**FSGS**	3631638	58914.30	1.38e-04	2380946	17795.25	1.38e-04
	FSSOR	21156	4104.17	1.36e-04	19153.0	3239.84	1.34e-05

CONCLUSION

For the time fractional diffusion problems, the paper presents the formulation of the Caputo's finite difference equations to generate a linear system. Then to solve the linear system, the formulation of FSGS and FSSOR iterative methods have been constructed based on the Caputo's derivative operator. From observation of all experimental results by imposing the FSGS and FSSOR iterative methods, it can be also observed in Table 1 and table 2 that the number of iterations and the execution time for FSSOR iterative method have been declined tremendously as compared with FSGS iterative method. This is due to the implementations of FSSOR iterative method have been accelerated by using the optimal value of the weighted parameter, ω. In fact, these conclusions are inline with the results of Othman and Abdullah (Ali, *et al.*, 2013). Based on their accuracy, it can be concluded that the numerical solutions for both methods are in good agreement.

Since this study has focused mainly on the full-sweep scheme, further observation of half-sweep (Abdullah, 1991; Yousif, *et al.*, 1995) and quarter-sweep (Othman, *et al.*, 2000); schemes needs to be carried out in solving to solve fractional diffusion equations. In addition to that, the capability of 4 Point-MEGSOR should also be investigated for solving other multi-dimensional fractional partial differential equations (Evans, 1985; Evans *et al.*, 1988) and being used as a smoother in multigrid solvers (Hackbusch, 1995; Othman, *et al.*, 2000). Also, discovery on various point block iterative methods can be also studied (Yousif, *et al.*, 1995; Martins, *et al.*, 2002) to solve the fractional problems.

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