SOLVING VARIABLE COEFFICIENT FOURTH-ORDER PARABOLIC EQUATION BY MODIFIED ADOMIAN DECOMPOSITION METHOD

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ABSTRACT. *In this work, a non-homogeneous variable coefficient fourth-order parabolic partial differential equation is solved by using the Adomian decomposition method (ADM). The ADM yields an explicit solution in the form of series which converges rapidly. The accuracy of ADM is also determined numerically. The modified ADM shows that the exact solution can be obtained by using four iterations only.*

KEYWORDS. *Adomian decomposition method, parabolic equations*

INTRODUCTION

One of the more recent methods of solving linear and non-linear equations of physics is the so-called Adomian decomposition method (ADM) (Adomian, 1988). The ADM which is requiring no transformation, linearization, perturbation or discretization, yields an analytical solution in the form of a rapidly convergent infinite power series with easily computable terms. There are three types of ADM such as standard ADM, modified ADM and two-steps ADM that have been proposed to solve numerous differential and integral equations like Burger-Fishers equation, Thomas-Fermi equations, Volterra and Fredholm integral equations and so on.

Besides ADM, there are also other numerical analytical methods that have been used to solve fourth-order boundary value problems such as finite difference, B-spline, homotopy perturbation method and variational iteration method (Noor & Mohyud-Din, 2006).

The variable coefficient fourth-order parabolic partial differential equations in one space variable arise in the study of the transverse vibrations of a uniform flexible beam. Numerical computations of the transverse vibrations have been carried out by a number of authors (Khaliq & Twizell, 1987). For instance, Evans (1965) who expressed the fourth-order parabolic equation in two space variables as a system of two second-order parabolic equations to be solved by finite difference methods while Khaliq and Twizell (1987) solved the problem of variable coefficient fourth-order parabolic partial differential equations by using a family of second-order methods.

Suzelawati *et al*. (2007) studied the numerical comparisons of AGE method and ADM in solving fourth-order parabolic equation. In that research, the non-homogeneous equation with constant coefficient was solved by using standard and modified ADM. As a continuation from the previous research, this paper will consider the non-homogeneous variable coefficient fourth-order parabolic equation, which is solved by using standard and modified ADM.

Lets consider the fourth-order parabolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2}+\mu(x)\frac{\partial^4 u}{\partial x^4}=f(x,t), \quad 0\le x\le 1, \quad t>0, \tag{1.1}$$

with the initial condition

$$u(x, 0)=g_0(x), \tag{1.2}$$

$$\frac{\partial u}{\partial t}(x, 0)=g_1(x), \tag{1.3}$$

and the boundary conditions at $x=0$ and $x=1$ are of the forms

$$u(0, t)=f_0(t), \quad u(1, t)=f_1(t), \tag{1.4}$$

and

$$\frac{\partial^2 u}{\partial x^2}(0, t)=p_0(t), \quad \frac{\partial^2 u}{\partial x^2}(1, t)=p_1(t). \tag{1.5}$$

METHOD

In the ADM, (Adomian, 1988), equation (1.1) is written in the form,

$$L_t u=f(x, t)-L_x u, \tag{2.1}$$

where L_t and L_x are the differential operators

$$L_t=\frac{\partial^2}{\partial t^2}, \quad L_x=\frac{\partial^4}{\partial x^4}. \tag{2.2}$$

Operating with $L_t^{-1}(\cdot)=\int_0^t\int_0^r p(\tau)\,\mathrm{d}\tau\,\mathrm{d}r$ on both sides of (2.1) gives

$$L_t^{-1}L_t u=L_t^{-1}(f(x, t))-L_t^{-1}L_x u, \tag{2.3}$$

or

$$u(x, t)=g_0(x)+tg_1(x)+L_t^{-1}(f(x, t))-L_t^{-1}L_x u. \tag{2.4}$$

The decomposition method consists of expressing the solution in a series form

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \tag{2.5}$$

Now substituting (2.5) into (2.4) gives

$$\sum_{n=0}^{\infty} u_n(x, t) = g_0(x) + tg_1(x) + L_t^{-1}(f(x, t)) - L_t^{-1} L_x\left(\sum_{n=0}^{\infty} u_n(x, t)\right). \tag{2.6}$$

Thus, each term of the series (2.5) can be determined recursively as (Adomian, 1988)

$$u_0 = g_0(x) + tg_1(x) + L_t^{-1}(f(x, t)), \tag{2.7}$$

$$u_{n+1} = -L_t^{-1} L_x u_n, \quad n \geq 0. \tag{2.8}$$

The exact solution in equations (2.7) and (2.8) is determined by

$$u(x, t) = \lim_{n \to \infty} \phi_n, \tag{2.9}$$

where

$$\phi_n = \sum_{k=0}^{n-1} u_k. \tag{2.10}$$

In modified ADM, equation (2.7) is assumed to be divided into two parts namely h_0 and h_1,

$$u_0 = h_0 + h_1 \tag{2.11}$$

The standard ADM with the recursive formula in (2.7) and (2.8) now become (Wazwaz, 1999)

$$u_0 = h_0, \tag{2.12}$$

$$u_1 = h_1 - L_t^{-1} L_x u_0, \tag{2.13}$$

$$u_{n+2} = -L_t^{-1} L_x u_{n+1}, \quad n \geq 0 \tag{2.14}$$

RESULT

Now consider the fourth-order parabolic problem as considered in (Wazwaz, 2001),

$$\frac{\partial^2 u}{\partial t^2} + (1+x)\frac{\partial^4 u}{\partial x^4} = \left(x^4 + x^3 - \frac{6}{7!}x^7\right)\cos t, \quad 0 < x < 1, \quad t > 0, \tag{3.1}$$

with the initial conditions

$$u(x, 0) = \frac{6}{7!}x^7, \quad 0 < x < 1, \tag{3.2}$$

$$\frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 < x < 1, \tag{3.3}$$

and the boundary conditions

$$\begin{aligned} &u(0, t) = 0, \quad u(1, t) = \frac{6}{7!}\cos t, \quad t > 0, \\ &\frac{\partial^2}{\partial x^2}u(0, t) = 0, \quad \frac{\partial^2}{\partial x^2}u(1, t) = \frac{1}{20}\cos t, \quad t > 0, \end{aligned} \tag{3.4}$$

The theoretical solution is

$$u(x, t) = \frac{6}{7!}x^7 \cos t. \tag{3.5}$$

Applying the recursive formula (2.7) and (2.8) yields

$$\begin{aligned} u_0 &= \frac{6}{7!}x^7 - \left(x^4 + x^3 - \frac{6}{7!}x^7\right)\cos t + \left(x^4 + x^3 - \frac{6}{7!}x^7\right) \\ &= \left(x^4 + x^3\right)(1 - \cos t) + \frac{6}{7!}x^7 \cos t \end{aligned} \tag{3.6}$$

$$u_1 = -\left(x^4 + x^3\right)(1 - \cos t) - 24(1+x)\left(\frac{t^2}{2!} - 1 + \cos t\right) \tag{3.7}$$

$$u_2 = 24(1+x)\left(\frac{t^2}{2!} - 1 + \cos t\right) \tag{3.8}$$

and so on.

According to Adomian & Rach (1992), the phenomena of the noise terms occurs only when solving non-homogeneous partial differential equations. In that study, it was shown that if a term in the component u_0 is cancelled by a term in u_1 even though u_1 introduces further terms, then the remaining non-cancelled terms in u_0 provide the exact solution. However, the phenomena is not always true for every non-homogeneous equations, and the non-homogeneity condition is not sufficient (Wazwaz, 1997).

It is obvious that the noise terms appear in components u_0 and u_1 that are given by

$$u_0 = \left(x^4 + x^3\right)\left(1 - \cos t\right) \tag{3.9}$$

and

$$u_1 = -\left(x^4 + x^3\right)\left(1 - \cos t\right) \tag{3.10}$$

Thus, by eliminating (3.9) and (3.10) gives the remaining non-cancelled terms in u_0 that is exactly same as (3.5).

By using the modified ADM in equations (2.12)-(2.14) gives

$$u_0 = \left(x^4 + x^3\right)\left(1 - \cos t\right) + \frac{6}{7!}x^7 \cos t \tag{3.11}$$

The terms u_0 in equations (3.11) will be split into two parts as

$$h_0 = \frac{6}{7!}x^7 \cos t \tag{3.12}$$

$$h_1 = \left(x^4 + x^3\right)\left(1 - \cos t\right) \tag{3.13}$$

By applying equations (2.12)-(2.14) gives

$$u_0 = \frac{6}{7!}x^7 \cos t, \tag{3.14}$$

$$\begin{aligned} u_1 &= x^4 - x^4 \cos t + x^3 - x^3 \cos t - x^3 + x^3 \cos t \\ &= x^4 - x^4 \cos t \end{aligned} \tag{3.15}$$

$$u_2 = 24 - 12t^2 - 24 \cos t, \tag{3.16}$$

$$u_{n+3} = 0, \quad n \geq 0. \tag{3.17}$$

The exact solution in equation (3.5) can be obtained by using modified ADM in four iterations only.

The accuracy of ADM is determined numerically for the approximate solution ϕ_2 as given in Table 1.

Table 1. Absolute errors for approximate solutions, ϕ_2 and theoretical solution, u_e

			Absolute Errors
t	x	u_e	ϕ_2
0.1	0.1	0.1184528768E-09	0.1099601185E-03
	0.2	0.1516196824E-07	0.1199551620E-03
	0.3	0.2590564417E-06	0.1299590564E-03
	0.4	0.1940731934E-05	0.1399507319E-03
	0.5	0.9254131002E-05	0.1499541310E-03
0.2	0.1	0.1166745925E-09	0.1757650117E-02
	0.2	0.1493434785E-07	0.1917434934E-02
	0.3	0.2551673340E-06	0.2077225167E-02
	0.4	0.1911596526E-05	0.2237011597E-02
	0.5	0.9115202546E-05	0.2396805203E-02
0.3	0.1	0.1137305344E-09	0.8883310114E-02
	0.2	0.1455750841E-07	0.9690884558E-02
	0.3	0.2487286788E-06	0.1049845873E-01
	0.4	0.1863361076E-05	0.1130603336E-01
	0.5	0.8885198001E-05	0.1211360520E-01
0.4	0.1	0.1096501183E-09	0.2801024011E-01
	0.2	0.1403521515E-07	0.3055662404E-01
	0.3	0.2398048088E-06	0.3310300980E-01
	0.4	0.1796507539E-05	0.3564939651E-01
	0.5	0.8566415495E-05	0.3819578642E-01
0.5	0.1	0.1044741145E-09	0.6817963010E-01
	0.2	0.1337268666E-07	0.7437778337E-01
	0.3	0.2284848885E-06	0.8057592848E-01
	0.4	0.1711703893E-05	0.8677408170E-01
	0.5	0.8162040196E-05	0.9297223204E-01

The numerical results for the approximate solution of problem (3.1) subject to conditions (3.2)-(3.4) by using the ADM and the exact solution are graphically the same as shown Figure 1.

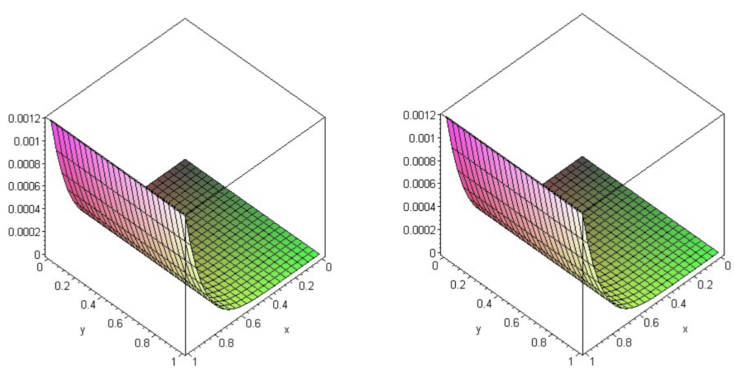

Figure 1. Exact versus 3-term approximant $\phi_3(x,t)$.

The $[p/q]$ Padé approximant to a function $f(x)$ is a polynomial of degree p divided by a polynomial of degree q which is chosen so that the leading terms of the power series of the approximant match the first $(p+q+1)$ terms of the power series of $f(x)$ (Boyd, 1997).

Using the Maple built-in Padé approximant [3,3] on ϕ_3 gives

$$\phi_{3[3,3]} = \frac{12x^7 - 5x^7 t^2}{840\left(12 + t^2\right)} \tag{3.18}$$

The combined Adomian-Padé approach yields generally better accuracy as compared with the theoretical solution, see Table 2.

Table 2. Numerical comparisons for $x = 0.5$

t	u_e	ADM, $\phi_{3[3,3]}$	Absolute Errors
0.125	0.9228028899E-05	0.9228028825E-05	0.74000000E-13
0.375	0.8654274757E-05	0.8654221824E-05	0.52933000E-10
0.50	0.8162040196E-05	0.8161746842E-05	0.29335400E-09
0.875	0.5961652327E-05	0.5953747013E-05	0.79053140E-08
1.00	0.5025133053E-05	0.5008012820E-05	0.17120233E-07

CONCLUSIONS

In this work, the ADM was applied to the solutions of a non-homogeneous variable coefficient fourth-order parabolic partial differential equation. The phenomena of noise terms is useful in demonstrating a fast convergence of the exact solution (i.e. by cancelling the same terms that occur in u_0 and u_1 will give the exact solution). The numerical results demonstrated that the ADM is accurate, reliable and requires less computation. The modified ADM applied in the problem considered in this work gives the exact solution in four iterations only.

For future works, research can be done by considering the homogeneous or nonlinear variable coefficient fourth-order parabolic partial differential equation. Besides that, this study needs to be conducted to solve fourth-order parabolic equation with constant coefficient by using ADM for Problem 1 as given by Khaliq and Twizell (1987).

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