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A REVIEW ON HOLOMORPHIC FUNCTIONS OF BOUNDED MEAN OSCILLATION

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ABSTRACT. *In this paper, we shall concentrate on developing the theory of holomorphic functions of bounded mean oscillation (BMOA). We give an alternative way to get a large subclass of BMOA with a proof different from that in Cima and Petersen (1976).*

KEYWORDS. Bounded Mean Oscillation.

INTRODUCTION

Functions of Bounded Mean Oscillation (BMO) were ûrst studied by John and Nirenberg (1961). The class of such functions is known to be Banach space, we shall denote this space by BMO and call a function in this space a BMO function in this paper. John and Nirenberg found a basic characterization of BMO functions in terms of their distribution function. Later developments have shown that BMO has close connections with other branches of analysis, such as harmonic analysis and complex analysis, and characterizations involving Poisson integrals and Carleson measures were obtained. The most spectacular result is perhaps Charles Fefferman's characterization of BMO as the dual space of the Hardy space $H^1(D)$. An important subspace of BMO consisting of functions of vanishing mean oscillation (VMO) was introduced by Donald Sarason in 1975.

In this paper, we shall concentrate on developing the theory of holomorphic functions of bounded mean oscillation (hereafter BMOA). We give in theorem an alternative way to get a large subclass of BMOA with a proof different from that in Cima and Petersen (1976).

BASIC NOTATIONS AND DEFINITIONS

Definition: *D* and *T*
D denotes the open unit disc and *T* denotes the unit circle.

Definition: Fourier coefficient $\hat{f}$
We associate to every $f \in L^1(T)$ a function $\hat{f}$ on ¢ defined by

$$\hat{f}(n) = \int_T f(t)e^{-int}\,d\sigma(t), \qquad n \in \mathbb{Z}.$$

Definition: Radial Limit f^*

Let f be a function defined on D. Define the radial limit of f by

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}).$$

It is well known that if $u \in L^p(T)$, $1 \le p \le \infty$, then the radial limit of $f = P[u]$ exists almost everywhere and $f^* = u$ a.e.

Definition: The space *H(D)*

H(D) is the class of all holomorphic functions in D.

Definition: The space $H^p(D)$

$H^p(D)$ is the class of all holomorphic functions in *D*. If f is any continuous function with domain *D*, we define f_r on T by

$$f_r(e^{i\theta}) = f(re^{i\theta}) \quad , \quad 0 \le r < 1$$

If $f \in H(D)$ and $0 \le p \le \infty$, we put

(a) $$\|f\|_p = \sup_{0 \le r < 1}\left\{\int_T |f(rz)|^p\, ds(z)\right\}^{\frac{1}{p}} \text{ where } 0 < p < \infty.$$

(b) $$\|f\|_0 = \sup_{0 \le r < 1}\left\{\int_T \log^+|f(r\zeta)|\,d\sigma(\zeta)\right\}.$$

(c) $$\|f\|_\infty = \sup_{z \in D}|f(z)|.$$

If $0 < p \le \infty$, $H^p(D)$ is defined as the class of all $f \in H(D)$ for which $\|f\|_p < \infty$.

Notation:

By *C(f)* and *C(p)* we mean a constant which may depend on the function f or the number p. We shall use C to denote a constant which may be different in each occurrence.

Definition: BMO

Let $f \in L^1(\sigma)$. For each arc I of T, let $I(f) = \dfrac{1}{\sigma(I)}\int_I f\,d\sigma$, the average of f over I, and let

$$\|f\|_* = \sup_I \frac{1}{\sigma(I)}\int_I |f - I(f)|\,d\sigma.$$

If $\|f\|_* < \infty$, we say that f is of bounded mean oscillation.

Let $BMO = \{f \in L^1(\sigma) : \|f\|_* < \infty\}$.

Definition: BMOA
We say that $f \in BMOA$ if $f \in H^1(D)$ and $f^* \in BMO$.
Alternatively, $f \in BMOA$ if $f \in BMO$ and $P[f] \in H(D)$.

RESULTS

Before we perform our proof, we need the following theorems.

Theorem (Hardy's inequality)

If $f(z)=\sum_{n=0}^{\infty} a_n z^n \in H^1(D)$, then $\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \le c\|f\|_1$.

Theorem (Paley's inequality)

If $\frac{\lambda_{n+1}}{\lambda_n} \ge q > 1$, then there exists a constant $C(q)$ such that for all $f(z)=\sum_{n=0}^{\infty} a_n z^n \in H^1(D)$,

$$\sum_{n=0}^{\infty} \left|a_{\lambda_n}\right|^2 \le C(q)^2 \|f\|_1^2 .$$

Now, we give an alternative way to get a large subclass of BMOA with a proof different from that in Cima and Peterson (1976).

Theorem
If $b=(b_1, b_2, b_3, \ldots) \in l^\infty$, then

$$\sum_{n=1}^{\infty} \frac{b_n}{n} z^n \in BMOA \quad \text{and} \quad \left\| \sum_{n=1}^{\infty} \frac{b_n}{n} z_n \right\|_* \le C\|b\|_\infty .$$

Proof:
Let μ be the measure on ¥ defined by

$$\mu(n)=\begin{cases} 1/n, & n \ne 0 \\ 0 \ , & n=0 \end{cases}.$$

Then Hardy's inequality shows that if $T: H^1(D) \to L^1(\mu)$ is defined by $Tf=\hat{f}$, then T is bounded.

By Fefferman's duality theorem, the adjoint of T is given by T^*: $l^\infty \to BMOA$. Since T is bounded, so is T^*, hence there is a $C<\infty$ such that

$$\|T^{*}b\|_{*} \leq C\|b\|_{\infty} \qquad \text{for all } b \in l^{\infty}. \tag{1}$$

Moreover,

$$< f, T^{*}b >=< Tf, b >= \sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n} \tag{2}$$

for $f(z) = \sum_{n=0}^{\infty} a_n z_n$ and for all $b \in l^{\infty}$. Let $T^{*}b = g$. We consider the case $f(z) = z^n$. From (2), we see that

$$\int_{T} e^{\text{int}} \overline{g}(t) d\sigma(t) = \frac{\overline{b_n}}{n}.$$

So $\hat{g}(n) = \frac{b_n}{n}, n \in ¥$. Thus

$$T^{*}b = g(z) = \sum_{n=1}^{\infty} \frac{b_n}{n} z^n. \tag{3}$$

By (1) and (3), we finally get

$$\left\| \sum_{n=1}^{\infty} \frac{b_n}{n} z^n \right\|_{*} \leq C\|b\|_{\infty} \text{ and } \sum_{n=1}^{\infty} \frac{b_n}{n} z_n \in BMOA \text{ for all } b \in l^{\infty}.$$

To see if the converse is true, we need the following theorem.

Theorem

If $\frac{\lambda_{n+1}}{\lambda_n} \geq q > 1$, then there exists a constant $C(q)$ such that

$$\left\| \sum_{n=0}^{\infty} c_n z_{\lambda_n} \right\|_{*} \leq C(q) \left\{ \sum_{n=0}^{\infty} |c_n|^2 \right\}^{\frac{1}{2}}.$$

Proof:

Let μ be the measure on ¥ such that

$$\mu(n) = \begin{cases} 1, & n = \lambda_k \\ 0, & \text{otherwise} \end{cases}.$$

Let $T : H^1(D) \to L^2(\mu)$ be defined by $Tf = \hat{f}$. Then Paley's inequality shows that T is bounded. Hence by Fefferman's duality theorem, the adjoint of T. i.e. $T^{*} : L^2 \to BMOA$, is bounded. Hence there is a $C(q) < \infty$ such that

$$\|T^{*}b\| \leq C(q)\|b\|_2, \qquad \forall b \in L^2(\mu). \tag{4}$$

Moreover,

$$< f, T^{*}b >=< Tf, b >= \sum_{k=0}^{\infty} a_{\lambda_k} \overline{b}_{\lambda_k},$$

$$\forall f(z)=\sum_{n=0}^{\infty} a_n z^n \in H^1(D) \text{ and } \forall b \in L^2(\mu). \qquad (5)$$

Let $T^* b = g$. We consider the case $f(z)=z^n$. Now (5) shows that

$$\int_T e^{int}\bar{g}(t)d\sigma(t)=\begin{cases}\bar{b}_{\lambda_k}, & n=\lambda_k\\ 0, & \text{otherwise}\end{cases}.$$

So

$$\hat{g}(n)=\begin{cases}\bar{b}_n, & n=\lambda_k\\ 0, & \text{otherwise}\end{cases}.$$

Thus

$$T^* b = g(z)=\sum_{k=0}^{\infty} b_{\lambda_k} z^{\lambda_k}.$$

$$=\sum_{k=0}^{\infty} c_k z^{\lambda k}, \text{ where } c_k = b_{\lambda_k}. \qquad (6)$$

By (4) and (6),

$$\left\|\sum_{n=0}^{\infty} c_n z^{\lambda_n}\right\|_* \le C(q)\left\{\sum_{n=0}^{\infty}|c_n|^2\right\}^{\frac{1}{2}}.$$

The following example shows that the converse of the above theorem is not true.

Example

Let

$$a_n=\begin{cases}\sqrt{2}^k, & n=2^k\\ 0, & \text{otherwise}\end{cases}.$$

Then $a=(a_1,a_2,a_3,\ldots)\notin l^{\infty}$, but

$$\sum_{n=1}^{\infty}\frac{a_n}{n}\in BMOA.$$

Proof:
Define

$$f(z)=\sum_{n=1}^{\infty}\frac{a_n}{n}z^n .$$

Then

$$f(z)=\sum_{k=1}^{\infty}\frac{\sqrt{2}^k}{2^k}z^{2k}=\sum_{k=1}^{\infty}\frac{\sqrt{2}^k}{2^k}z^{\lambda_k}\text{, where } \lambda_k=2^k .$$

Since $\frac{\lambda_{k+1}}{\lambda_k}=2>1$, by the above theorem,

$$\begin{aligned}\left\|\sum_{n=1}^{\infty}\frac{a_n}{n}z^n\right\|_* &= \left\|\sum_{k=1}^{\infty}\frac{\sqrt{2}^k}{2^k}z^{2^k}\right\|_* \\ &= \left\|\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k z^{2^k}\right\|_* \\ &\le C(2)\left\{\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^{2k}\right\}^{\frac{1}{2}} \\ &= C(2)\left\{\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\right\}^{\frac{1}{2}} \\ &= C(2).\end{aligned}$$

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