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A NOTE ON THE DYNAMICS IN A BILLIARD SYSTEM

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ABSTRACT. *In this note we investigate the motion of a particle bouncing inside a billiard. The dynamics of the billiard system is illustrated by the phase space in terms of s (the position) and p (the tangential momentum). A certain computer algebra system is used to generate these phase spaces. Different types of billiard are shown to generate different orbits. Four types of billiard have been investigated which are the circular, stadia, elliptical and oval billiards. Circular and elliptical billiards are shown to generate the regular motions. In the stadia-billiard, the motion of particle displayed chaotic behaviour. For an oval-billiard, it is observed that both regular and irregular motions of the particle existed.*

KEYWORDS. Chaos, closed orbit, regularity, phase space

INTRODUCTION

Dynamics is the study of the motions of objects and the forces that cause them. The developments in dynamics aimed at understanding the behaviour of systems over long times. The basic rules for predicting the behaviour of systems are developed by Newton in the seventeenth century. Sometimes the behaviour of systems is predictable, but some systems are found to exhibit chaos and this eventually resulted in breakdown of predictability.

The dynamics in billiard system has been investigated, in order to illustrate the features of regularity and chaos in dynamical systems. This billiard problem discussed the motion of a particle bouncing inside the billiard, and this beautiful idea was introduced originally by Berry (1981). Further researches are widespread. For example, Koiller et al (1996) have suggested billiards with moving boundaries as the limiting case of rigid bodies. Kirillov (1996) found unexpected application of the billiard system in cosmological problems. Beletsky and Pankova (1996) used the billiard problem to describe the interaction of two mass points connected by an non-extensible weightless thread whereby the center of mass of which moves along the circular Kepler orbit. A review on the current state of this research field is given by Bunimovich (2000).

In this note, we will consider the billiard as a region of the plane bounded by a closed curve *C*. The particle is moving freely in this region and making elastic collisions with the enclosure. Assuming the motion of particle is in a straight line, we have simple reflection at each bounce and no dissipation. The particle bounces in accordance with the conservation law, i.e. its angle of reflection equals the angle of incidence. Bouncing map of the particle is specified by the evolution of position and tangential momentum from the collision inside the billiard. This construct of the billiard system turns out to be instructive, with dynamics depending particularly on the shape of the boundary.

This note is organized as follows. We will introduce the model of the billiard mapping and notion of stable closed orbits. The four types of billiard (circular, stadia, elliptical and oval) are then discussed and their phase space trajectories depicted using certain computer algebra (for example, *Mapleã* and *Mathematicaã*).

MODEL OF BILLIARD MAPPING

The orbit of a particle bouncing inside the billiard may be specified by the sequence of its positions and tangential momentums after each impact. The position around the curve C is parametrised by either the arc length s or by the direction ψ, where ψ is the angle measured from the origin (anticlockwise-pointing), as shown in Figure 1.

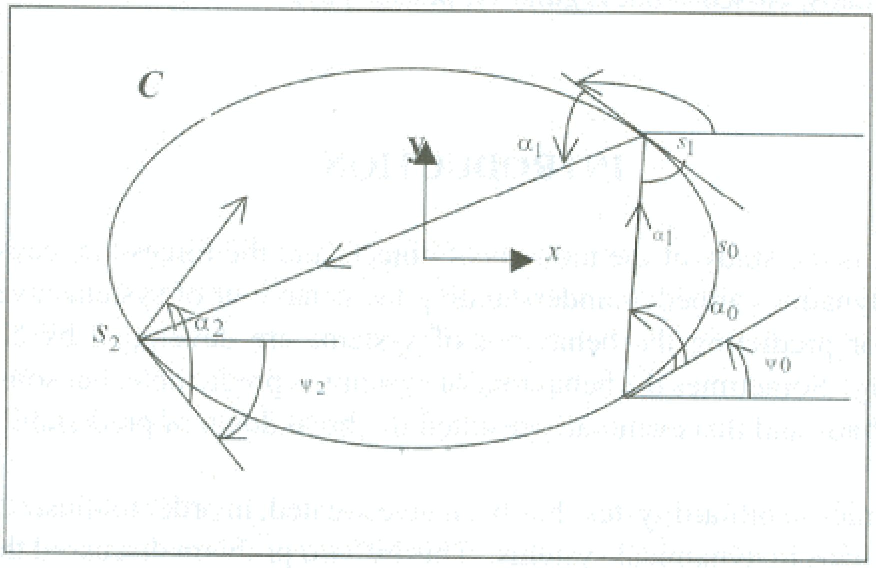


Figure 1. Billiard geometry

Closed curve C can be defined by giving its radius of curvature R as a function of ψ, therefore the two parameters s and ψ can be related,

$$R(\psi) = \frac{ds}{d\psi},$$

$$\int_{\psi} ds = \int_{\psi} R(\psi)\,d\psi, \qquad (\psi_0 = \pi/2 \text{ and } \psi' \text{ is a dummy})$$

$$s(\psi) = \int_{\pi/2}^{\psi} d\psi' R(\psi'). \tag{1}$$

The direction of the orbit after impact will be labeled by its angle α with respect to the forward tangent, or by the tangential momentum p, defined by

$$p \equiv \cos\alpha. \tag{2}$$

Since the angles ψ and α can be measured directly from the graph (see Figure1), so the ψ, α description is more convenient for calculating the orbits. For theoretical purposes the s, p description is preferable.

This discrete dynamics is a mapping M of the phase space with coordinates s, p and is symbolized by

$$\begin{pmatrix} s_{n+1} \\ p_{n+1} \end{pmatrix} = M \begin{pmatrix} s_n \\ p_n \end{pmatrix}.$$

The bounce mapping, M is usually non-linear. M is area preserving, and in terms of the variables s, p, i.e.

$$\frac{\partial(s_1, p_1)}{\partial(s_0, p_0)} = \mathbf{det} \begin{bmatrix} \frac{\partial s_1}{\partial s_0} & \frac{\partial s_1}{\partial p_0} \\ \frac{\partial p_1}{\partial s_0} & \frac{\partial p_1}{\partial p_0} \end{bmatrix} = 1. \tag{4}$$

In order to show that the billiard mapping M is area preserving, we have to evaluate the derivatives in equation (4). Referring to Figure 2, it is clear that in consequence of small initial deviations $\delta s_0, \delta\alpha_0$ the deviation δs_1 is given by

$$\delta s_0 \,\mathbf{sin}\,\alpha_0 + \delta s_1 \,\mathbf{sin}\,\alpha_1 = \rho_{01}(\delta s_0 + \delta\psi_0). \tag{5}$$

The angles are related by

$$\delta\alpha_0 + \delta\psi_0 = \delta\psi_1 - \delta\alpha_1 \tag{6}$$

To obtain these relations in terms of s and p, we invoked equations (1) and (2), and after some manipulations,

$$\begin{pmatrix} \delta s_1 \\ \delta p_1 \end{pmatrix} = \begin{pmatrix} \delta s_1/\delta s_0 & \delta s_1/\delta p_0 \\ \delta p_1/\delta s_0 & \delta p_1/\delta p_0 \end{pmatrix} \begin{pmatrix} \delta s_0 \\ \delta p_0 \end{pmatrix} \equiv m_{1,0} \begin{pmatrix} \delta s_0 \\ \delta p_0 \end{pmatrix} \tag{7}$$

and the deviation matrix $m_{1,0}$ is

$$m_{1,0} = \begin{bmatrix} \dfrac{-\sin\alpha_0}{\sin\alpha_1} + \dfrac{\rho_{01}}{\sin\alpha_1 R(\psi_0)} & \dfrac{-\rho_{01}}{\sin\alpha_0 \sin\alpha_1} \\ \dfrac{-\rho_{01}}{R(\psi_0)R(\psi_1)} + \dfrac{\sin\alpha_1}{R(\psi_0)} + \dfrac{\sin\alpha_0}{R(\psi_1)} & \dfrac{-\sin\alpha_1}{\sin\alpha_0} + \dfrac{\rho_{01}}{\sin\alpha_0 R(\psi_1)} \end{bmatrix}. \tag{8}$$

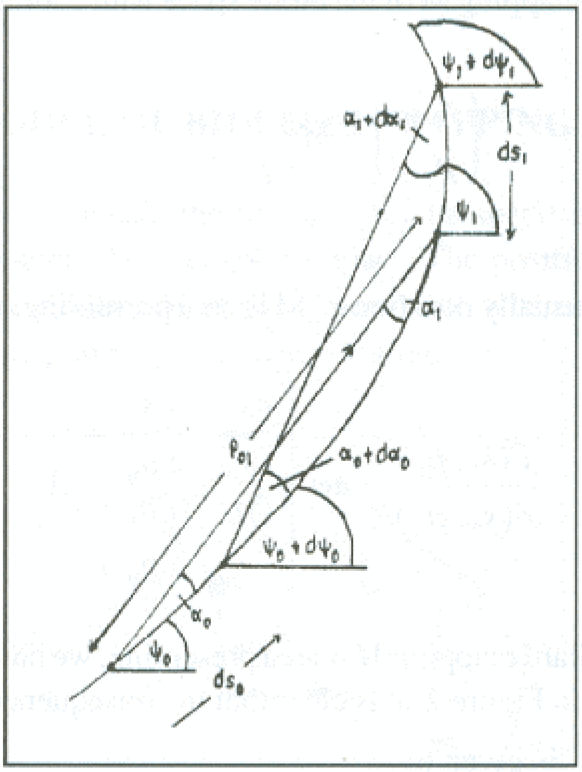


Figure 2. Geometry of deviation

After N bounces, deviations accumulate by successive multiplication of matrices of the form (8). In particular, for an N-bounce closed orbit the matrix m_N is given in terms of the bounce geometry by

$$m_N = m_{N,N-1} m_{N-1,N-2} \ldots m_{3,2} m_{2,1} m_{1,0}. \tag{9}$$

There are three possibilities in which the orbit can be generated by the iterations of M. These orbits can be explored in phase space and will be considered in the later section. The three possibilities included:

1) A finite set of N points $s_0, p_0; s_1, p_1; \ldots; s_{N-1}, p_{N-1}$ may be encountered repeatedly, (close after N bounces). Such a closed orbit satisfies

$$\begin{pmatrix} s_{n+N} \\ p_{n+N} \end{pmatrix} = M^N \begin{pmatrix} s_n \\ p_n \end{pmatrix} = \begin{pmatrix} s_n \\ p_n \end{pmatrix}. \tag{10}$$

2) The iterations of s_0, p_0 may fill a smooth curve in phase space, the whole curve maps onto itself under *M*.

3) The iterations of may fill an area in phase space. This will happens when the orbit, evolves in a chaotic manner whose detail is sensitively dependent on the values of and .

With reference to Figure 1, the mapping equations considered before can be found in terms of ψ, α. The slope of the trajectory segment beginning ψ_0, α_0 at is given by the quotient of *x* and *y* increments around the curve between ψ_0 and ψ_1. Therefore two mapping equations will be generated as below

$$\left(\int_{\psi_0}^{\psi_1} R(\psi)\sin\psi d\psi\right)\left(\int_{\psi_0}^{\psi_1} R(\psi)\cos\psi d\psi\right)^{-1} = \tan(\psi_0 + \alpha_0) \tag{11}$$

and

$$\alpha_1 = \psi_1 - \psi_0 - \alpha_0. \tag{12}$$

STABILITY OF CLOSED ORBIT

The closed orbits, which satisfy equation (10), may be stable or unstable. When s_0 and p_0 have returned to their starting point, after *N* iterations, the deviations δs_N and δp_N of the nearby orbit will be

$$\begin{pmatrix}\delta s_N \\ \delta p_N\end{pmatrix} = m_N \begin{pmatrix}\delta s_0 \\ \delta p_0\end{pmatrix}. \tag{13}$$

The stability of orbits depends on the eigenvalues of m_N, i.e. $\lambda_\pm$. The eigenvalues, $\lambda_\pm$, $\lambda_\pm = \frac{1}{2}\left\{Trm_N \pm \left[(Trm_N)^2 - 4\right]^{1/2}\right\}$ are given in terms of the trace of m_N by the following.

(14)

where

$$Trm_N = -\left[\frac{\sin\alpha_1}{\sin\alpha_0} + \frac{\sin\alpha_0}{\sin\alpha_1} - \frac{\rho_{01}}{\sin\alpha_0 R(\psi_1)} - \frac{\rho_{01}}{\sin\alpha_1 R(\psi_0)}\right].$$

There are three possibilities in determining the stability of an orbit:

(i) If $|Trm_N| < 2$, it follows from (14) that $\lambda_\pm$ are complex conjugates $\lambda_\pm^j = e^{\pm ij\beta}$.

If j increases, $\lambda_\pm^j$ will approach to zero and remain bounded, therefore the orbit is stable.

(ii) If $|Trm_N| > 2$, it follows from (14) that are real , the positive exponent show that almost all deviations grow exponentially, therefore the orbit is unstable.

(iii) If both eigenvalues are +1 or −1. The deviations grow linearly; and thus in this case the orbit has neutral stability.

For the diametrical two-bounce orbit with impacts on the opposite sides of C at normal incidence ($\alpha_0 = \alpha_1 = \frac{1}{2}\pi$, $p_0 = p_1 = 0$), if the radii of curvature R are the same at both impacts, and if the length of each trajectory segment is ρ, then it follows from (8) and (9) that the stability conditions are:

$$\frac{\rho}{2R} - 1 \begin{cases} > 0 & \textit{instability}, \\ < 0 & \textit{stability}. \end{cases} \tag{15}$$

Now we consider the motion of a particle moving in the circular billiard. When C is a circle, the radius of curvature $R(\psi)$ is independent of ψ. In order to find the sequence of the positions (s) and tangential momentums (p) after each impact, mapping equations will be used to find and then followed by the conversion to s and p. The procedure is as follows. Based on equations (11) and (12), and since $R(\psi)$ is a constant,

$$\cos\psi_1 \cos(\psi_0 + \alpha_0) + \sin\psi_1 \sin(\psi_0 + \alpha_0) = \cos\psi_0 \cos(\psi_0 + \alpha_0) + \sin\psi_0 \sin(\psi_0 + \alpha_0),$$

$$\psi_1 = \psi_0 + 2\alpha_0.$$

From (12),

$$\alpha_1 = \psi_1 - \psi_0 - \alpha_0.$$

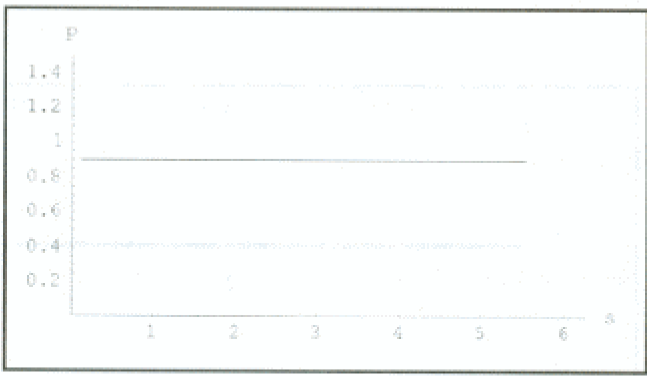
The conversion to s, p terms, according to (1) and (2), are

$$s_1 = R(\psi)(\psi_1 - \psi_0) + s_0,$$

$$p_1 = \cos\alpha_1.$$

When α is an irrational submultiple of π, an orbit will never repeats but continually hits C at different points s_n. This will eventually filling an annulus within C. In phase space, the iterations of s, p will fill the invariant curve $p = \cos\alpha_1$ as in Figure 3.

When α, is a rational submultiple of π, in the phase space the iterations of s, p repeatedly return to N points on the line $p = \cos\pi K/N$. For example if $\alpha = 3\pi/5$, the iterations of s, p will return to the same place after five bounces (as shown in Figure 4). Therefore this orbit will close after five times of impact. It can be shown that the closed orbits in a circular billiard have neutral stability.



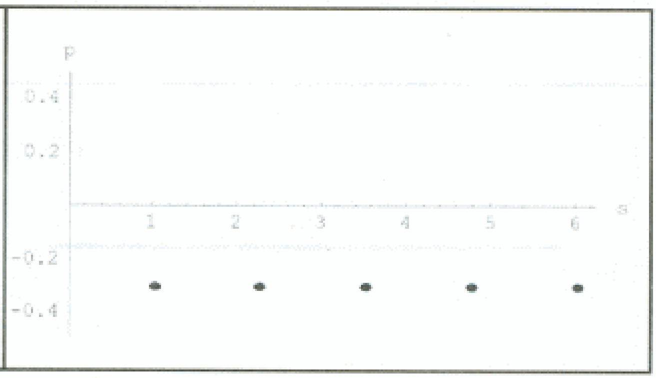


Figure 3. Phase space trajectory for orbit with α is an irrational submultiple of

Figure 4. Phase space trajectory for orbit generated by $\alpha = 3\pi/5$.

STADIA-BILLIARD

Bunimovich (1974) has proved that the stadia is ergodic in nature. Ergodic theory is a statistical study of complex dynamical system first proposed by Maxwell, Boltzman, Gibbs, and Poincaré (example Arnold & Avez, 1968). In general terms, ergodic theory can be thought of as understanding the behaviour of typical orbits.

To illustrate that the stadia is ergodic, we generate the computation as follows. From (11) and (12), we obtain

$$\psi_1 = 2\alpha_0 + \psi_0 + \frac{\eta}{100},$$

$$\alpha_1 = \psi_1 - \psi_0 - \alpha_0,$$

$$s_1 = \psi_1 - \psi_0 + s_0,$$

$$p_1 = \cos\alpha_1.$$

Iterations of 900 to 5000 times of the above computations with different η, followed by plotting the phase spaces will yield the results as shown in Figure 5. The ergodic billiard will cause the mapping of *s, p* come arbitrarily close to every point in phase space.

There exists a family of two-bounce non-isolated closed orbits formed by perpendicular impacts on the straight section of *C*. The existence of the two holes in Figure 5(d) is connected with the family of two-bounces non-isolated closed orbits. The phenomenon of resonance indicates that the amplitude is a maximum when the frequencies are the same, where $\omega_0 = \omega_1$ (example Kibble & Berkshire, 1996). In our case, when the frequency of straight line impacts with $p = 0$, is almost the same with the frequency of impact with δp, the orbit will resonate, resulting the holes in phase space. It can be shown that this long closed two-bounce orbit is unstable.

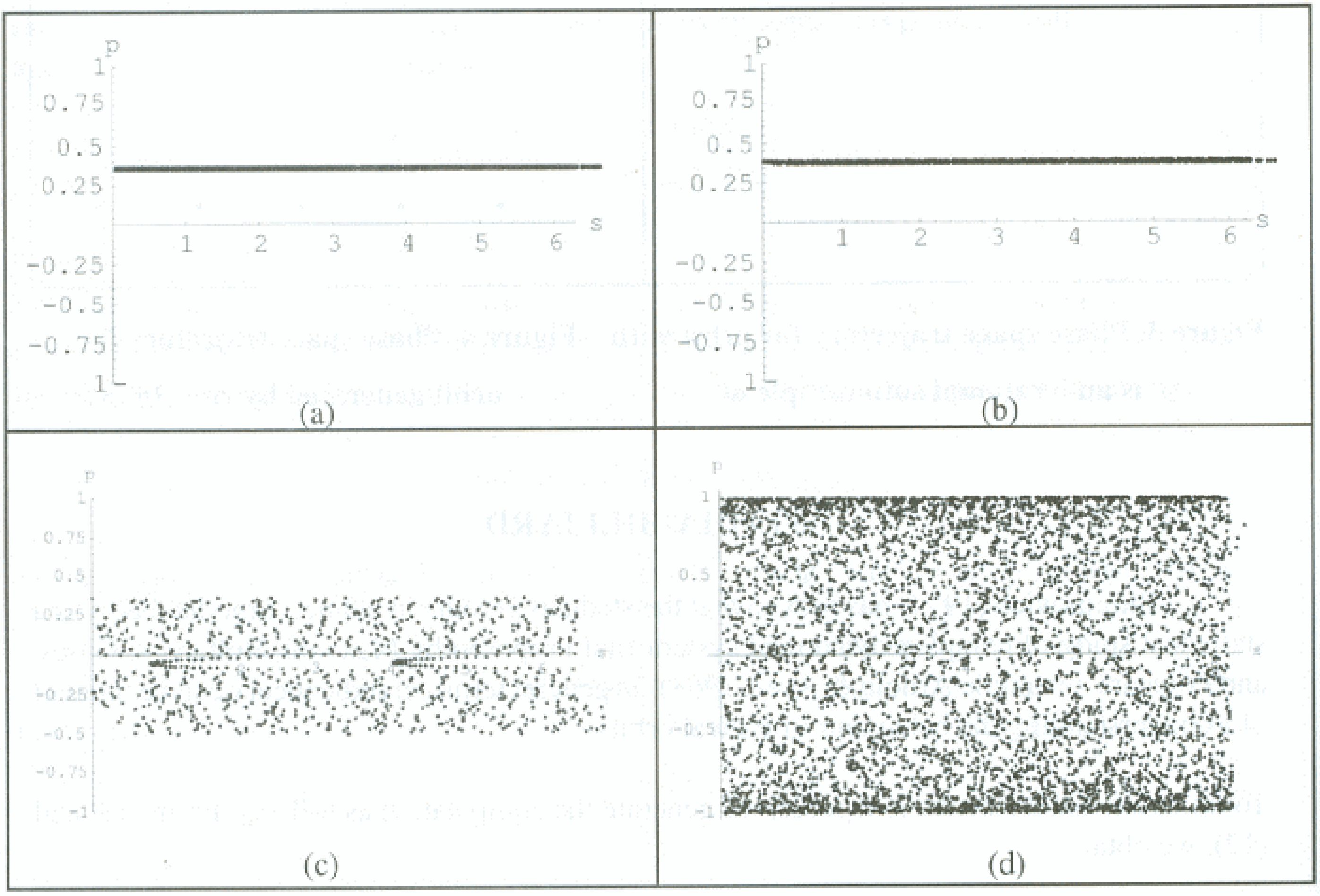


Figure 5. Phase spaces of stadia-billiard for (a), (b), (c) with η = 0.001, 0.01, 0.1 respectively with 900 bounces; and (d) η = 1 with 5000 bounces.

ELLIPTICAL BILLIARD

An ellipse is a sort of elongated circle that is the intersection of a circular conical surface and a plane that cuts the surface in a single closed curve (see example James, 1992). The parametric equations of elliptical billiard are as follows (example Berry, 1981):

$$\begin{aligned} x &= a\cosh M \cos\lambda\ , \\ y &= a\sinh M \sin\lambda. \end{aligned} \tag{16}$$

The foci lie at $x = \pm a, y = 0$ and radius is

$$R(\psi) = \frac{a\cosh M \sinh M}{\left(\cosh^2 M \sin^2\psi + \sinh^2 M \cos^2\psi\right)^{3/2}} \tag{17}$$

the eccentricity e of the ellipse is

$$e = \left(\cosh M\right)^{-1}. \tag{18}$$

The parameter λ is related to the direction ψ by

$$\begin{aligned} \tan\psi &= \frac{dy}{dx} \\ &= -\tanh M \cot\lambda\ . \end{aligned} \tag{19}$$

Using the equations (11), (16), (17), (18) and (19), we generate a cont in Figure 6. Actually this plot illustrates two kinds of orbit. There are orbits, which bounce all around C, exploring all values of s whilst repeatedly touching an ellipse, and there are orbits, which bounce across C and exploring a restricted range of s whilst repeatedly touching a hyperbola (Loo, 2001). There are some invariant curves as shown in the figure. Along some of these invariant curves, motion of particle will be regular (periodic) as in the case of the circular billiard and neutral stability is assumed.

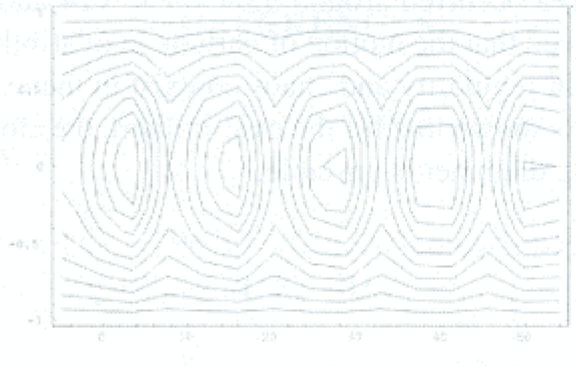


Figure 6. Contour of elliptical billiard mapping.

OVAL-BILLIARD

Let us consider the simplest deformation of a circle into an oval is assumed to be the following form (example Kibble & Berkshire, 1996)

$$R(\psi) = a + a\delta\cos 2\psi = a(1+\delta\cos 2\psi). \tag{20}$$

We can determine the Cartesian coordinates as follows.

$$x(\psi) = \mathrm{a}\left[\left(1+\frac{1}{2}\delta\right)\sin\psi + \frac{1}{6}\delta\sin 3\psi\right],$$

$$y(\psi) = \mathrm{a}\left[\left(-1+\frac{1}{2}\delta\right)\cos\psi - \frac{1}{6}\delta\cos 3\psi\right]. \tag{21}$$

To plot a phase space trajectory of a particle bouncing inside an oval-billiard (see Loo, 2001), we start from equation (11), together with (20) and (21), and obtain

$$\cos(\psi_0+\alpha_0-\psi_1) - \frac{1}{2}\delta\cos(\psi_0+\alpha_0+\psi_1) + \frac{1}{6}\delta\cos(3\psi_1-\psi_0-\alpha_0)$$
$$= \cos(\alpha_0) - \frac{1}{2}\delta\cos(2\psi_0+\alpha_0) + \frac{1}{6}\delta\cos(2\psi_0-\alpha_0).$$

Some notion of the richness of orbital structure for generic billiards can be obtained from Figure 7 which portrays the magnification of oval billiard mapping with d = 0.3 under 600 of iterations. Resolving this equation required high precision: it was necessary to solve the mapping equation (11) for ψ, to one part in 2^{25} (as observed by Berry, 1981). From Figure 7, we can observe that the chaotic area occurred around $3.09 < s < 3.24$ and the other area filled with invariant curves. This showed that the motion of particle in oval-billiard displayed the generic property since the feature of regularity and chaos co-exist in the oval-billiard. From relations (15), (20) and (21), it can be shown that for the oval-billiard, the closed orbit of short diameter is stable and whilst for long diameter is unstable.

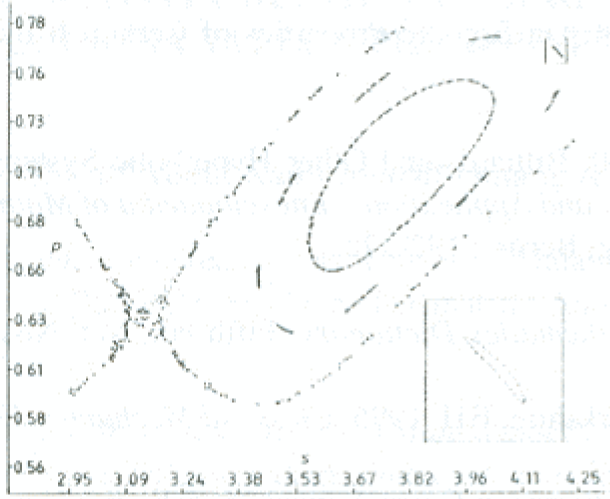


Figure 7. Magnification of oval billiard mapping under 600 of iterations (inset: further magnification of right-most minute formation).

CONCLUSION

In this note, we have investigated four types of billiard, which are the circular, stadia, elliptical and oval billiards, via certain computer algebra systems. Depending on the shape of the closed curve C, this billiard system is observed to exhibit very different behaviours. When the curve C (billiard) is circular, the s-p space is covered with invariant curves (an invariant curve in phase space means that the motion of particle is regular). When the shape of the billiard is a stadia, almost all orbits explore almost all s, p values. Consequently this resulted in the unpredictable (chaotic) motions. Elliptical billiard, instead generated motions entirely confined to invariant curves. While oval billiard rendered motions in which the phase space is filled with chaotic areas as well as covered with invariant curves. Finally, we have illustrated in this note the co-existing behaviours of regularity and chaos in classical mechanics via a fundamental example i.e. the billiard system.

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