# COMPARISON BETWEEN DIRECT SAMPLING AND IMPORTANCE SAMPLING METHOD FOR THE ALOHA MODEL

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**ABSTRACT.** *The main aim of this paper is to compare the performance between the direct sampling method and importance sampling method. The slotted ALOHA model (which is the simplest form of broadcast network) was considered. The communication channel is a given radio frequency and a packet is transmitted by broadcasting it. For the slotted ALOHA system, transmission will only start at regularly spaced points in time, say the integer points (0, 1, 2,...). Thus if a packet arrives sometime during the interval (n – 1, n) it will start the transmission at time n and will complete it at time n + 1. That transmission will be successful if no other packets are transmitted at the same time. However, if there exists packets transmitted by other users at the same instant, collisions will occur. When a collision occurs, the senders back off and attempt to retransmit their packets after random periods of time. There are two main points of interest namely (i) the probability that the number of blocked users reaches saturation point, that is, the system becomes unstable and (ii) the expected duration until the system becomes unstable. Two types of simulation method were used, that is, direct sampling method and importance sampling method. The importance sampling method is a variance reduction technique and is used to speed up the simulation of the rare events. The simulation results show that the probability that the process reaches saturation point is very small. Thus the occurrence of this process is very rare indeed. A statistical analysis of the simulation results showed that by using importance sampling variance is reduced.*

**KEYWORDS.** Importance sampling, Rare events, ALOHA model, Variance reduction technique.

## INTRODUCTION

The performance of the direct sampling method and the importance sampling method is being studied via the slotted ALOHA model (which is the simplest form of a broadcast network). The ALOHA model was developed by Abramson (1970, 1977). A better model known as the slotted ALOHA model was developed by Roberts (1973). For this method, transmissions start at regularly spaced points in time, say (0, 1, 2, ...). Thus if a packet arrives sometime during the interval ($n - 1, n$), it will start transmission at time $n$ and will complete at time $n + 1$. That transmission will be successful if no other packets are transmitted at the same time. However when there are more than one transmission, collisions will occur. If the number of collision were large, the system becomes unstable. Thus, we are interested in the probability that the system becomes unstable. This probability is very, very small. Thus it is a rare event.

Simulation is required to obtain the probability that the system becomes unstable. Direct simulation can be used but it would require a very large amount of computation and imposes severe demands on the pseudo-random number generator. An alternative method is to use the importance sampling method, which is a variance reduction technique. For the slotted ALOHA model, Cottrell, Fort and Malgouyres (1983) used the large deviations technique as a method to speed up the simulation. We compare Cottrell's method with the direct sampling method in terms of speed and the reduction in variance achieved.

## THE MATHEMATICAL MODEL

We assume that new packets (whose transmission has not been attempted before) arrive in a Poisson stream with rate $\alpha$. Packets that have suffered at least one collision and have not yet been transmitted are said to be 'blocked'. While the packet remains blocked, it makes an attempt to transmit in each subsequent slot with a fixed probability, p.

The state of the system at the beginning of the t[th] slot is described by a single integer, N(t), which represents the number of blocked packets at that moment. Assuming that N(t) is a Poisson process with parameter , then N(t) is a transient homogenous Markov chain (N. Abramson, 1977 ) over **N** for every values of and p. Let the one-step transition probability (i0, j0) be $p_{ij}$ = P{( N(t + 1) = j ½ N(t) = i)}. Suppose that the system is in state i at time t, that is, there are i blocked packets. The state at time n + 1 will depend on what happens during the interval (n, n+1). The possibilities are:-

1. If (j – i) 2, the packets will collide and the chain will jump to state j.

2. If one new packet arrives during the n[th] th slot and at least one of the blocked packets attempt to transmit, then again there will be collision and the next state is i + 1.

3. If either one new packet arrives and no blocked packets attempt to transmit (in which case the new arrival is successful), or no new packets arrive and any number other than one of the blocked packets attempt to transmit, then the state will remain i.

4. If no new packets arrive and exactly one of the blocked packets attempts to transmit, then the attempt will be successful and the next state is i – 1.

Now the probability that there are j–i arrivals during a slot is given by $\alpha^{j-i}e^{-\alpha} / (j-i)!$. Hence, we obtain

$$p_{ij} = \begin{cases} 0 & \text{for } j \le i-2 \\ ip(1-p)^{i-1}e^{-\alpha} & \text{for } j = i-1 \\ (1-p)\alpha e^{-\alpha} + \left[1 - ip(1-p)^{i-1}\right]e^{-\alpha} & \text{for } j = 1 \\ \alpha e^{-\alpha}\left[1-(1-p)^{i}\right] & \text{for } j = i+1 \\ \alpha^{j-i}e^{-\alpha} / (j-i)! & \text{for } j \ge i+2 \end{cases}$$

Here the one-step transition is from state i to state j and the chain makes a jump of size j – i. The average jump size when the chain is in state i is called the drift at state i. For any given i, the drift is given by $b_i$ = E[ N(t + 1) – N(t)½N(t) = i ]. For suitable choices of $\alpha$ (in particular for $\alpha\tau < 1/e$), we can see that N(t) remains for a very long time period near a value $n_o$ and if it reaches (or overtakes) a value $n_c$ (critical point corresponding to the channel saturation), it will go to infinity. These points $n_o$ and $n_c$ are the mean equilibrium points, $n_o$ is a stable point while $n_c$ is an unstable point. These points are obtained by observing the changes in sign of E[ N(t + 1) – N(t)½N(t) = i ] between i and i + 1. So b(i) < 0 for $n_o < i < n_c$ and b(i) > 0 for 0 i $n_o$ .

Now let $\beta$ = Pr (A) = Pr (starting from $n_o$ the process comes to $n_c$ without returning to $n_o$ ). The jump size at state i does not vary too much as a function of i . Also the jumps are small. Thus the process N(t) can be described as slow Markov walk. Now consider the trajectory of N(t). Every trajectory of N(t) starting from $n_o$ can be cut into sections. We call the sections that come back to $n_o$ as sections of the first kind. The sections that reach $n_c$ are called the sections of the second kind. Therefore the number of sections in a trajectory before the exit is a geometric random variable with parameter $\beta$ and mean value $1/\beta$ . Now let $\tau_0$ be the common expection of the durations of the sections of the first kind. Let $\tau$ be the mean exit time of a process which starts from $n_o$.

Then

$$\tau \approx \tau_0 / \beta$$

The approximation is because we ignore the duration of the last section of the trajectory. Now $\tau$ can be estimated by simulating N(t) with transition probability $p_{ij}$ and taking the average to get $\tau$.

## SIMULATION METHOD

To estimate , we have used the direct simulation method. This method calculates P(A) just by using he original values of $p_{ij}$. The algorithm below gives the required method. Let A be the event 'starting from $n_o$, the process comes directly to $n_c$ without returning to $n_o$'.

### Algorithm 1

1. Input $\alpha$ and p.
2. Get $n_o$ and $n_c$.
3. Calculate $\beta$ directly.
4. Return $\tau = \tau_0 / \beta$.

## IMPORTANCE SAMPLING METHOD

The importance sampling method used in this paper was developed by Cottrell, Fort and Malgouyres (1983). The transformed probabilities below was used to simulate n sections $w_1$, $w_2$, ....., $w_n$.

$$\tilde{p}_{ij} = \begin{cases} e^{\lambda_i (j-i)} p_{ij} & \text{for } n_0 < i < n_c, \lambda_i > 0 \text{ and } E[e^{\lambda_i (j-i)}] = 1 \\ p_{ij} & \text{for } i \le n_0 \text{ or } i \ge n_c \end{cases}$$

Also, $$\frac{dP}{d\tilde{P}}(w_i) = \exp\left( -\sum_t \lambda_{w_i(t)} [w_i(t+1) - w_i(t)] \right).$$

Finally, $$\beta = \frac{1}{n} \sum_{i=1}^{n} 1_A(w_i) \frac{dP}{d\tilde{P}}(w_i)$$

Hence algorithm 2 was obtained.

**Algorithm 2**

1. Input $\alpha$ and p.
2. Get $n_o$ and $n_c$.
3. Get $\tau_o$.
4. Get $$\tilde{p}_{ij} = \begin{cases} e^{\lambda_i (j-i)} p_{ij} & \text{for } n_0 < i < n_c, \lambda_i > 0 \text{ and } E[e^{\lambda_i (j-i)}] = 1 \\ p_{ij} & \text{for } i \le n_0 \text{ or } i \ge n_c \end{cases}$$
5. Calculate $$\frac{dP}{d\tilde{P}}(w_i) = \exp\left( -\sum_t \lambda_{w_i(t)} \left[ w_i(t+1) - w_i(t) \right] \right).$$
6. Calculate $$\beta = \frac{1}{n} \sum_{i=1}^{n} 1_A(w_i) \frac{dP}{d\tilde{P}}(w_i)$$
7. Return $$\tau = {}^{\tau_0}\!/\!_{\beta}.$$

For each simulation we used 15, 000, 000 iterations and five replicates were used. The pseudo-random numbers were generated using the congruential generator given by Marsaglia (1987) (See also Ripley. 1987). The computer used for this simulation is the SUN 4 / 330.

## RESULTS

Tables 1, 2, 3 and 4 below give the means, standard deviations and times per run in minutes for the direct sampling method and the importance sampling method respectively. These were obtained for (a) $\alpha = 0.30$, $p = 0.049$ and (b) $\alpha = 0.27$, $p = 0.064$.

**Table 1. Results from the direct sampling method when $\alpha$ = 0.30, p = 0.049**

| $\beta(\times 10^{-4})$ | Mean of $\beta(\times 10^{-4})$ | Std Dev of $\beta(\times 10^{-7})$ | Mean run time of $\beta$(mins) | $\tau(\times 10^{-4})$ | Mean of $\tau(\times 10^{-4})$ |
|---|---|---|---|---|---|
| 1.45<br>1.37<br>1.42<br>1.40<br>1.43 | 1.41 | 28.9 | 11.52 | 5.88<br>6.11<br>6.04<br>6.02<br>6.00 | 6.01 |

**Table 2. Results from the importance sampling method when $\alpha$ = 0.30, p = 0.049**

| $\beta(\times10^{-4})$ | Mean of $\beta(\times10^{-4})$ | Std Dev of $\beta(\times10^{-7})$ | Mean run time of $\beta$(mins) | $\tau(\times10^{-4})$ | Mean of $\tau(\times10^{-4})$ |
|---|---|---|---|---|---|
| 1.39 | | | | 6.13 | |
| 1.39 | | | | 6.03 | |
| 1.39 | 1.39 | 2.36 | 28.60 | 6.18 | 6.11 |
| 1.39 | | | | 6.06 | |
| 1.39 | | | | 6.16 | |

**Table 3. Results from the direct sampling method when $\alpha$ = 0.27, p = 0.064**

| $\beta(\times10^{-5})$ | Mean of $\beta(\times10^{-5})$ | Std Dev of $\beta(\times10^{-8})$ | Mean run time of $\beta$(mins) | $\tau(\times10^{-5})$ | Mean of $\tau(\times10^{-5})$ |
|---|---|---|---|---|---|
| 1.97 | | | | 2.98 | |
| 2.04 | | | | 2.87 | |
| 2.05 | 2.01 | 37.8 | 15.50 | 2.86 | 2.92 |
| 2.01 | | | | 2.92 | |
| 1.98 | | | | 2.97 | |

**Table 4. Results from the importance sampling method when $\alpha$ = 0.27, p = 0.064**

| $\beta(\times10^{-5})$ | Mean of $\beta(\times10^{-5})$ | Std Dev of $\beta(\times10^{-8})$ | Mean run time of $\beta$(mins) | $\tau(\times10^{-5})$ | Mean of $\tau(\times10^{-5})$ |
|---|---|---|---|---|---|
| 2.06 | | | | 2.85 | |
| 2.05 | | | | 2.37 | |
| 2.06 | 2.05 | 4.67 | 18.60 | 2.87 | 2.86 |
| 2.06 | | | | 2.85 | |
| 2.06 | | | | 2.86 | |

The results show that variance is considerably reduced using importance sampling method than the direct sampling method. However, the importance sampling method takes a little longer than the direct sampling method.

## CONCLUSION

This paper looks at the slotted ALOHA model. Our aim is to compare the performance between the direct sampling method and the importance sampling method by obtaining the probability that the number of blocked packets reaches saturation point, that is, when the system becomes unstable and to obtain its expected duration. The result shows that this probability is very small indeed.

Direct sampling can be used to simulate such rare events. However, a huge amount of pseudo-random numbers are required. Thus importance sampling which is an example of a variance reduction technique can be used. This method will increase the reliability of the estimates as seen by the smaller variance. The results obtained show that variance have been reduced considerably (up to approximately 8 times) when using importance sampling. However, the time taken for each process is increased slightly.

## REFERENCES

Abramson, N. 1970. The ALOHA System – Another Alternative For Computer Communications. *Proc. For The Fall Joint Computer Conference*.

Abramson, N. 1977 The Troughput of Packet Broadcasting, *IEEE Trans. Commun.*, Vol. **Comm-25**.

Cottrell, M., Fort, J. C. and Malgouyres, G. 1983. Large Deviations And Rare Events In The Study Of Stochastic Algorithms, *IEEE Trans. Auto. Control*, Vol. **AC-28**.

Marsaglia, G. 1987. *The Structure Of Linear Congruential Sequences*, In Applications Of Number Theory To Numerical Analysis, Z.K.Zaremba (Ed.). New York: Academic Press.

Ripley, B. D. 1987. *Stochastic Simulation*. New York: John Wiley.

Roberts, L. 1973. Dynamic Allocation of Satellite Capacity Through Packet Reservation, *Proc NCC*.