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AN ANALYSIS FOR THE ERROR OF A EULER'S FORMULA IN THE INNER BOUNDARY-LAYER FROM AN UNSTEADY FREE CONVENCIVE FROM A CIRCULAR CYLINDER

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ABSTRACT. *This article reveals an error in the application of the complex notation by the analytical solution i.e. the matched asymptotic expansion in the problem of free convection in Newtonian fluid for Prandtl number,* $P \neq 1$. *Even by convention that the solution is the real part of a complex quantity when adopting the complex notation, an extra real part will occur when taking the real part of the product of two complex quantities from the non-linear partial differential equations of the problem. However, the purpose of this article is not to give a full solution to the problem, but to identify the variation of the error from the correct expansion.*

KEYWORDS: Complex number, Euler formula, free convection.

INTRODUCTION

Analytical studies have been undertaken on oscillatory flows due to an oscillating temperature on the surface of a horizontal circular cylinder for unbounded region by Merkin (1967), Chatterjee and Debnath (1979) provided that Prandtl number, $P \neq 1$, while Roslan *et al.* (2003) for $P = 1$. Merkin (1967), Chatterjee and Debnath (1979) and Roslan *et al.* (2003) adopted the methods of Schlichting (1932) and Riley (1965) in the oscillatory flows, i.e. used a matching technique between the inner and outer boundary layers. The solution technique requires by dividing the domain outside the cylinder into two regions, an inner boundary-layer and an outer boundary-layer. It is found that at the outer edge of the inner boundary-layer a steady flow is induced. Merkin (1967) and Chatterjee and Debnath (1979) adopted the complex notation with the convention that the solution is the real part of a complex quantity. However, Roslan *et al.* (2003) adopted the use of the real part of any complex quantity in order to avoid an extra real part occurring when taking the real part of the product of two complex quantities.

The flow field is divided into two regions: an inner region (boundary-layer region) adjacent to the cylinder and an outer region far from the cylinder. Locally valid expansions of the temperature and stream function in terms of the small parameter $\varepsilon = 1/S$, where S is Strouhal number, are developed in both of these regions. The solutions for the stream function and the temperature in both of these regions is obtained up to second-order in ε, and more attention was given to the inner region from the second-order stream function and temperature, due to the appearance of the non-linear term in the second-order inner boundary-layer equations. The full solution in the two regions was obtained, see Roslan *et al.* (2004), However, for the purpose of this short note we will concentrate our discussion only in the inner boundary-layer to show that a direct implementation by the Euler's formula led the results of the second-order of temperature and stream function to an errors, and not to give a full solutions in the inner region and outer region. For convenience, Roslan's notation will be used where possible.

Governing equations

We consider a fixed horizontal circular cylinder of radius a in which the temperature of the cylinder surface, T' oscillates harmonically with a frequency ω and a amplitude bT'_∞ about a mean temperature, T'_∞, where b is the non-dimensional amplitude in the temperature oscillations. We take polar coordinates (r',θ) in which coordinate r' is defined as the distance measured outwards from origin of the cylinder with $r'=0$ at the axis of the cylinder and θ is defined to be anticlockwise angle made by the outward normal with the downward vertical from the origin of the cylinder with $\theta=0$ in the direction of gravity. Since in free convection problems the density variation is due to very small changes in the temperature, we assume that the Boussinesq approximation is applicable. The non-dimensional governing equations can be written as, see Roslan *et al.* (2004),

$$\frac{\partial(\nabla^2\psi)}{\partial t}-\frac{\varepsilon}{r}\frac{\partial(\nabla^2\psi,\psi)}{\partial(r,\theta)}=\varepsilon^2\gamma\nabla^4\psi+\frac{1}{r}\frac{\partial(r\cos\theta,T)}{\partial(r,\theta)}, \tag{1}$$

$$\frac{\partial T}{\partial t}-\frac{\varepsilon}{r}\frac{\partial(T,\psi)}{\partial(r,\theta)}=\frac{\varepsilon^2\gamma}{P}\nabla^2 T, \tag{2}$$

$$\text{where } \nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{3}$$

The boundary conditions, for all time, are as follows:

$$\left.\begin{aligned}&\psi=0,\quad \frac{\partial\psi}{\partial r}=0,\quad T=\cos t, && at\quad r=1,\quad 0\le\theta\le 2\pi\\ &\psi\to\text{constant},\quad \frac{\partial\psi}{\partial r}\to 0,\quad T\to 0, && as\quad r\to\infty,\ 0\le\theta\le 2\pi\end{aligned}\right\}, \tag{4}$$

In the equations (1 to 4), the non-dimensional variables are as follows:

$$\begin{aligned}&t=\omega t',\quad r=r'/a,\quad u=u'/U_c,\quad v=v'/U_c,\quad p=(p'-p'_\infty)/\rho U_c^2,\\ &\gamma=v\omega/U_c^2,\quad T=(T'-T'_\infty)/bT'_\infty,\quad \varepsilon=\frac{U_c}{a\omega},\quad P=\frac{v}{K},\end{aligned} \tag{5}$$

where t', v, p', p'_∞, ρ and K are the time, the kinematic viscosity, the pressure, the constant pressure at large distances from the cylinder, the density and the thermal conductivity of the fluid respectively. u' and v' are the fluid velocity components in the r' and θ directions, respectively. $P \neq 1$ is the Prandtl number, and the stream function ψ relates the fluid velocities components (u,v) in the (r,θ) coordinate system, in the usual manner $u = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$ and $v = \frac{\partial \psi}{\partial r}$. Here, we write γ in terms of Reynolds number, R_c and Strouhal number, S, in which $\gamma = \frac{1}{\varepsilon R_c} = \frac{S}{R_c}$ where $R_c = U_c a/v$, $\varepsilon = 1/S$ and $U_c = g\beta b T_\infty' / \omega$ is an appropriate velocity scale.

Inner boundary-layer equations

The asymptotic expansions for ψ and T in the inner region are assumed to be of the form:

$$\psi^{(i)}(t,r,\theta) = 0 + \varepsilon\left(\psi_0{}^{(i)}(t,r,\theta) + \varepsilon\psi_1{}^{(i)}(t,r,\theta) + \varepsilon^2\psi_2{}^{(i)}(t,r,\theta) + h.o.t\right), \tag{6}$$

$$T^{(i)}(t,r,\theta) = T_0{}^{(i)}(t,r,\theta) + \varepsilon T_1{}^{(i)}(t,r,\theta) + \varepsilon^2 T_2{}^{(i)}(t,r,\theta) + h.o.t\,. \tag{7}$$

First, the radial non-dimensional coordinate in the inner region is modified since it is advisable for it to assume the value zero at the surface of the cylinder where our non-dimensional cylinder radius is 1. Now we consider the solution in the very thin inner boundary-layer, thus we introduce the inner radial coordinate

$$R = \frac{r-1}{\varepsilon^a}\,. \tag{8}$$

Substitution of expansions (6) and (7) into equations (1) and (2) and equating in power of ε results in the following set of equations for $\psi_n^{(i)}$ and $T_n^{(i)}$ in the inner region, where $n = 0, 1, 2, \ldots$,

$$\frac{\partial T_0^{(i)}}{\partial t} = \frac{\gamma}{P}\frac{\partial^2 T_0^{(i)}}{\partial R^2}, \tag{9}$$

$$\frac{\partial^3 \psi_0^{(i)}}{\partial t \partial R^2} - \frac{\partial^4 \psi_0^{(i)}}{\partial R^4} = \sin\theta\,\frac{\partial^2 T_0^{(i)}}{\partial R^2}, \tag{10}$$

$$\frac{\partial T_1^{(i)}}{\partial t} - \frac{\gamma}{P}\frac{\partial^2 T_1^{(i)}}{\partial R^2} = \frac{\partial(T_0^{(i)},\psi_0^{(i)})}{\partial(R,\theta)} + \frac{\gamma}{P}\frac{\partial^2 T_0^{(i)}}{\partial R^2}, \tag{11}$$

$$\frac{\partial^3 \psi_1^{(i)}}{\partial t \partial R^2} - \gamma\frac{\partial^4 \psi_1^{(i)}}{\partial R^4} = \frac{\partial\left(\partial^2\psi_0^{(i)}/\partial R^2, \psi_0^{(i)}\right)}{\partial(R,\theta)} - \frac{\partial^2 \psi_0^{(i)}}{\partial t \partial R} + 2\gamma\frac{\partial^3 \psi_0^{(i)}}{\partial R^3} + \sin\theta\,\frac{\partial T_1^{(i)}}{\partial R} + \cos\theta\,\frac{\partial T_0^{(i)}}{\partial \theta}. \tag{12}$$

Solution without complex notation

Equations (9 to 12) are solved without applying the complex notation and the following terms are used to deal with the solutions when a couple between flow properties is appear in the boundary-layer equations,

$$\cos^2 t = \frac{1}{2}(1+\cos 2t), \quad \cos t \sin t = \frac{1}{2}(\sin 2t) \text{ and } \sin^2 t = \frac{1}{2}(1-\cos 2t). \tag{13}$$

By solving equations (9 to 12), subject to the boundary conditions (4) we obtain the solutions are as follows, see Roslan (2002),

$$T_0^{(i)} = e^{-\sqrt{P}\eta}\cos\left(t-\sqrt{P}\eta\right), \tag{14}$$

$$\begin{aligned}\psi_0^{(i)} &= \sqrt{\frac{\gamma}{2P}}\frac{1}{1-P}\sin\theta \ \cos t\left\{-\sqrt{P}e^{-\eta}(\cos\eta+\sin\eta)+e^{-\sqrt{P}\eta}\left(\cos\sqrt{P}\eta+\sin\sqrt{P}\eta\right)-1+\sqrt{P}\right\}\\ &+\sqrt{\frac{\gamma}{2P}}\frac{1}{1-P}\sin\theta \ \sin t\left\{\sqrt{P}e^{-\eta}(\cos\eta-\sin\eta)+e^{-\sqrt{P}\eta}\left(\sin\sqrt{P}\eta-\cos\sqrt{P}\eta\right)+1-\sqrt{P}\right\},\end{aligned} \tag{15}$$

$$\begin{aligned}T_1^{(i)} &= \cos\theta\left\{b_3e^{-\sqrt{2P}\eta}\cos\left(\sqrt{2P}+2t\right)+\frac{1}{4(1-P)}e^{-2\sqrt{P}\eta}\cos\left(2\sqrt{P}-2t\right)\right\}\\ &+\cos\theta\left\{\frac{1-\sqrt{P}}{2(1-P)}e^{-\sqrt{P}\eta}\cos\left(\sqrt{P}-2t\right)+d_5e^{-\left(1+\sqrt{P}\right)\eta}\cos\left(\left[1+\sqrt{P}\right]\eta-2t\right)\right\}\\ &+\frac{\cos\theta}{2}\left\{\frac{1}{1+\sqrt{P}}e^{-\sqrt{P}\eta}\sin\left(\sqrt{P}\eta\right)-\frac{P\sqrt{P}}{(1+P)^2}e^{-\left(1+\sqrt{P}\right)\eta}\sin\left(\sqrt{P}-1\right)\eta-\frac{1}{2(P-1)}e^{-2\sqrt{P}\eta}\right\}\\ &+\frac{\cos\theta}{2}\left\{\frac{2P^2}{(P-1)(1+P)^2}e^{-\left(1+\sqrt{P}\right)\eta}\cos\left(\sqrt{P}-1\right)\eta-\frac{1+3P}{2(1+P)^2}\right\}\\ &-\sqrt{\frac{\gamma}{2}}\eta e^{-\sqrt{P}\eta}\cos\left(\sqrt{P}\eta-t\right)+C\eta,\end{aligned} \tag{16}$$

$$\begin{aligned}\psi_{1\,(\text{steady})}^{(i)} &= -\sqrt{\frac{\gamma}{P^3}}\frac{\sin 2\theta}{4\sqrt{2}(1-P)^2}\left\{\frac{P^{3/2}}{2}e^{-2\eta}+\frac{1+P}{4}e^{-2\sqrt{P}\eta}+P(\sqrt{P}-1)e^{-\eta}(\cos\eta-\sin\eta)\right\}\\ &+\sqrt{\frac{\gamma}{P^3}}\frac{\sin 2\theta}{4\sqrt{2}(1-P)^2}\left\{P(1-\sqrt{P})e^{-\sqrt{P}\eta}\left(\cos\sqrt{P}\eta-\sin\sqrt{P}\eta\right)\right\}\\ &+N\sqrt{\frac{\gamma}{2}}\sin 2\theta e^{-\left(1+\sqrt{P}\right)\eta}\left\{L\cos\left(\sqrt{P}-1\right)\eta+M\cos\left(\sqrt{P}-1\right)\eta\right\}\\ &+2P\eta\sqrt{\frac{\gamma}{P^3}}\frac{\sin 2\theta}{4\sqrt{2}(1-P)^2}\left\{\left(1-\sqrt{P}\right)^2-\left[\frac{1+3P}{4\sqrt{P}}+\frac{\left(p^2-6P+1\right)\left(P^3+2P+1\right)}{(1+P)^4}\right]\right\}\end{aligned} \tag{17}$$

$$\left.\begin{aligned}
&\text{where } \eta = \frac{R}{\sqrt{2\gamma}},\ C = g_1 \cos\theta \quad \text{and } g_1 \text{ is an arbitrary constant,} \\
&d_5 = \frac{P\sqrt{P}\left[\left(1+\sqrt{P}\right)^2 + 2P\right]}{2(1-P)\left[4P^2 - \left(1+\sqrt{P}\right)^4\right]}, \quad b_3 = \left(\frac{3-2\sqrt{P}}{4(P-1)} - d_5\right)\cos\theta, \\
&L = 4\sqrt{P}(1-P)\left(1-\sqrt{P}\right) - \left(1+\sqrt{P}\right)\left(6P - P^2 - 1\right), \\
&M = -4\sqrt{P}(1-P)\left(1+\sqrt{P}\right) - \left(1-\sqrt{P}\right)\left(6P - P^2 - 1\right), \\
&N = \frac{P\sqrt{P}}{4(1-P)^5(1+P)} - \frac{1}{4\sqrt{P}(1+P)^3(1-P)^2}.
\end{aligned}\right\} \tag{18}$$

Here, we could consider for the steady part of the second-order of stream function.

Solutions by using complex notation

For convenience we use notation $\psi_{nc}^{(i)}$ and $T_{nc}^{(i)}$, $n = 0, 1, 2, \ldots$ to refer the solution of $\psi_n^{(i)}$ and $T_n^{(i)}$ by using complex notation, respectively. Thus, the solutions of the resulting equations (9 to 12) which satisfy the boundary conditions (4) provided $P \neq 1$, are as follows:

$$T_{0c}^{(i)} = e^{-\sqrt{P}(1+i)\eta} e^{it}, \tag{19}$$

$$\psi_{0c}^{(i)} = \sqrt{\frac{\gamma}{2P}} \cdot \frac{1+i}{P-1}\left\{\sqrt{P}\left(e^{-(1+i)\eta} - 1\right) - e^{-\sqrt{P}(1+i)\eta} + 1\right\}\sin\theta\, e^{it}, \tag{20}$$

$$\begin{aligned}
T_{1c}^{(i)} = {} & \frac{e^{2it}\cos\theta}{P-1}\left\{\left(\sqrt{P}-1\right)e^{-\sqrt{P}(1+i)\eta} - \frac{1}{2}e^{-2\sqrt{P}(1+i)\eta} + \beta_1 e^{-\sqrt{2P}(1+i)\eta} + \beta_2 e^{-\left(\sqrt{P}+1\right)(1+i)\eta}\right\} \\
& - \sqrt{\frac{\gamma}{2}}\eta e^{-\sqrt{P}(1+i)\eta} e^{it},
\end{aligned} \tag{21}$$

$$
\begin{aligned}
\psi_{1c}^{(i)} = {} & \frac{2\gamma e^{it}\sin\theta}{P-1}\left\{\frac{\sqrt{P}(2-\sqrt{\gamma})+(2\sqrt{\gamma}-2)}{4\sqrt{\gamma P}}e^{-(1+i)\eta}-\frac{1+i}{4}\eta e^{-(1+i)\eta}-\frac{1}{4P}e^{-\sqrt{P}(1+i)\eta}\right. \\
& \left.+\frac{1+i}{4\sqrt{P}}\eta e^{-\sqrt{P}(1+i)\eta}+\frac{\sqrt{P}-1}{\sqrt{\gamma P}}\cdot\frac{1+i}{2}\eta+\frac{\gamma(P-2\sqrt{P}+1)-2\sqrt{\gamma}(P-\sqrt{P})}{4\gamma P}\right\} \\
& +\sqrt{2\gamma}\frac{1+i}{4}\cdot\frac{e^{2it}\sin 2\theta}{P-1}\left\{\beta_3 e^{-(1+i)\eta}+\beta_4 e^{-\sqrt{P}(1+i)\eta}+\beta_5 e^{-2\sqrt{P}(1+i)\eta}+\beta_6 e^{-\sqrt{2P}(1+i)\eta}\right. \\
& \left.+\beta_7 e^{-\left(1+\sqrt{P}\right)(1+i)\eta}+\beta_8 e^{-\sqrt{2}(1+i)\eta}+\beta_9\right\},
\end{aligned}
\tag{22}
$$

$$
\left.
\begin{aligned}
& \text{where}\quad \beta_1=\frac{3+4\sqrt{P}-7P}{2\left(1+2\sqrt{P}-P\right)},\quad \beta_2=\frac{P\sqrt{P}}{1+2\sqrt{P}-P}, \\
& \beta_3=\frac{1-\sqrt{P}}{\sqrt{P}(P-1)},\quad \beta_4=\frac{\sqrt{P}-1}{\sqrt{P}(P-1)},\quad \beta_5=\frac{1}{8\sqrt{P}(1-2P)}, \\
& \beta_6=\frac{3-7P+4\sqrt{P}}{4\sqrt{2P}(P-1)(1-P+2\sqrt{P})},\quad \beta_7=\frac{\sqrt{P}(1+P-P^2)+(1-3P-P^2)}{\sqrt{P}\left(1+\sqrt{P}\right)^2(1-P+2\sqrt{P})(1-P-2\sqrt{P})}, \\
& \beta_8=-\frac{1}{\sqrt{2}}\left\{\beta_3+\sqrt{P}\beta_4+2\sqrt{P}\beta_5-\sqrt{2P}\beta_6+\left(1+\sqrt{P}\right)\beta_7\right\}\text{and } \beta_9=-(\beta_5+\beta_6+\beta_7+\beta_8).
\end{aligned}
\right\}
\tag{23}
$$

CONCLUSION

We found that for the first-order solution for the temperature and the stream function $T_{0c}^{(i)}$ and $\psi_{0c}^{(i)}$, are identical to the solution of $T_0^{(i)}$ and $\psi_0^{(i)}$, respectively. However, for the second-order solution for the temperature and the stream function, studying the real part of $T_{1c}^{(i)}$ and $\psi_{1c}^{(i)}$ are not identical with $T_1^{(i)}$ and $\psi_1^{(i)}$, respectively. These can be identified by employing $\eta\to\infty$ for the temperature and the tangential velocity for the second-order temperature and stream function $T_1^{(i)}$, $T_{1c}^{(i)}$, $\psi_1^{(i)}$ and $\psi_{1c}^{(i)}$. From equations (16) and (21) we obtain

$$
\lim_{\eta\to\infty} T_1^{(i)} = -\frac{1+3P}{4(1+P)^2}\cos\theta,\quad \lim_{\eta\to\infty} T_{1c}^{(i)} = 0,
\tag{24}
$$

while, from equations (17) and (22) we obtain

$$\lim_{\eta\to\infty}\frac{\partial\psi_1^{(i)}}{\partial\eta}_{(\text{steady})} = 2P\sqrt{\frac{\gamma}{P^3}}\frac{\sin 2\theta}{4\sqrt{2}(1-P)^2}\left\{\left(1-\sqrt{P}\right)^2 - \left[\frac{1+3P}{4\sqrt{P}} + \frac{\left(p^2-6P+1\right)\left(P^3+2P+1\right)}{(1+P)^4}\right]\right\},$$

$$\lim_{\eta\to\infty}\frac{\partial\psi_{1c}^{(i)}}{\partial\eta} = \frac{2\gamma e^{ii}\sin\theta}{P-1}\cdot\frac{\sqrt{P}-1}{\sqrt{\gamma P}}\cdot\frac{1+i}{2}, \tag{25}$$

where $\lim_{\eta\to\infty}\frac{\partial\psi_{1c}^{(i)}}{\partial\eta}_{(\text{steady})} = 0$.

Clearly, by the expressions (24) and (25), even Euler's formula is mathematically correct, however, it is not physically applicable in these problem. Further, the steady part from the solutions vanished in the complex notation compared to the use of the real part of any complex quantity. By adopting Euler's formula in the non-linear governing equations, an extra real part occurring when taking the real part of the product of two complex quantities. Thus, an appropriate step must be taken when we employ the complex notation as the form of the solution in our calculation.

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APPENDIX

For simply, we show the agreement above as follows:
Let ψ_a and ψ_b is unsteady stream function with time t, and coefficient A and B respectively,

$$\psi_a(t) = Ae^{it}, \tag{26}$$

$$\psi_b(t) = Be^{it}. \tag{27}$$

T_a is unsteady temperature with coefficient C, thus

$$T_a(t) = Ce^{it}. \tag{28}$$

A couple between $\psi_a\psi_b$ will gives us

$$\psi_a\psi_b = Ae^{it}Be^{it}. \tag{29}$$

By implementing Euler's formula we obtain

$$Ae^{it}Be^{it} = ABe^{2it} = AB(\cos 2t + i\sin 2t) \tag{30}$$

where the real part of $\psi_a\psi_b$ is given by

$$\mathrm{Re}(\psi_a\psi_b) = AB\cos 2t. \tag{31}$$

However, by considering the real part before the expressions are multiplied we obtain

$$\begin{aligned}\psi_a\psi_b &= \mathrm{Re}(Ae^{it})\cdot\mathrm{Re}(Be^{it})\\ &= AB\cdot\mathrm{Re}(e^{it})\cdot\mathrm{Re}(e^{it})\\ &= AB\cdot\mathrm{Re}(\cos t + i\sin t)\cdot\mathrm{Re}(\cos t + i\sin t)\\ &= AB\cdot\cos t\cdot\cos t\\ &= AB\cdot\cos^2 t\end{aligned} \tag{32}$$

where by applying expression (13) we obtain that

$$\mathrm{Re}(\psi_a\psi_b) = \frac{AB}{2} + \frac{AB\cos 2t}{2} \tag{33}$$

where the first term on the right hand side as a representative of the steady part from a couple between $\psi_a\psi_b$. With the same argument we can work for $T_a(t)$.

By comparing equations (31) and (33) we note that some imaginary part have becomes a real part when we multiplying e^{it} twice or where the non-linear term are exist.

$$e^{2it} = e^{it}\cdot e^{it} = (\cos^2 t + ii\sin^2 t) + i(2\sin t\cos t) \tag{34}$$

By considering the real part from equation (34) by using expression (13) obtain

$$(\cos^2 t + ii\sin^2 t) = \cos^2 t - \sin^2 t = \cos 2t. \tag{35}$$

Thus we conclude that a couple $\psi_a\psi_b$ or $\psi_a T_a$ directly will give us an extra term which are stated above. Further the steady part is vanish, has been shows by expressions (33) and (34).