GENERAL DIFFERENTIAL TRANSFORMATION METHOD FOR HIGHER ORDER OF LINEAR BOUNDARY VALUE PROBLEM

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ABSTRACT. *In this paper, we propose the generalization of differential transformation method to solve higher order of linear boundary value problem. Previous studies show that the differential transformation method is a powerful method to solve several lower order linear boundary value problems. In our study, we generalized the method so that one can solve n-th order boundary value problems with m-th order linear differential equation for $m > n, m < n$ or $m = n$. To illustrate the accuracy of the proposed method, we provide several numerical examples and we compare the results with the exact solutions. The comparisons demonstrate the proposed method has high accuracy.*

KEYWORDS. Differential transformation method; boundary value problems; linear differential equations.

INTRODUCTION

The differential transformation method (DTM) is one of the numerical methods in ordinary differential equations, partial differential equations and integral equations. Since proposed in (Zhou, 1986), there are tremendous interests on the applications of the DTM to solve various scientific problems. For instance, see (Arikoglu & Ozkol, 2005), (Ayaz, 2004), (Bildik *et al.* 2006), (Chen & Ho, 1999) and (Chen & Liu, 1998). One of the problems that solvable by this method is the boundary value problems (BVPs). This can be observed in (Jang & Chen, 1997), (Erturk & Momani, 2007), (Abdel-Halim Hassan & Erturk, 2009) and (Islam *et al.*, 2009). Previous studies concluded that the DTM can be easily applied in linear and nonlinear differential equations.

The DTM is developed based on the Taylor series expansion. This method constructs an analytical solution in the form of polynomial.

Definition 1

A Taylor polynomial of degree n is defined as follows:

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!}\left(f^{(k)}(c)\right)(x-c)^k \quad (1)$$

Theorem 1

Suppose that the function f has $(n+1)$ derivatives on the interval $(c-r, c+r)$, for some $r > 0$ and $lim_{x\to\infty} R_n(x) = 0$, for all $x \in (c-r, c+r)$ where $R_n(x)$ is the error between $P_n(x)$

and the approximated function $f(x)$. Then, the Taylor series expanded about $x = c$ converges to $f(x)$.
That is,

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(f^{(k)}(c)\right)(x-c)^k \tag{2}$$

for all $x \in (c-r, c+r)$.
The DTM is an approximation to exact solution of the functions which are differentiable in the form of polynomials. This method is different with the traditional high order Taylor series since the series needs more time in computation and requires the computation of the necessary derivatives (Odibat *et al*., 2009). The DTM can be applied in the high order differential equations and it is alternative way to get Taylor series solution for the given differential equations. This method finally gives series solution but truncated series solution in practice. In addition, the series of the method always coincides with the Taylor expansion of exact solution because it has very small error. In (Ayaz, 2003), Ayaz was studied in application of two-dimensional DTM in partial differential equations. On the other hand, Borhanifar and Abazari were studied for two-dimensional and three-dimensional DTM in partial differential equations in (Borhanifar & Abazari, 2009).

In terms of application, the first use of DTM is in solution of initial value problems in electric circuit analysis which conducted by Zhou (Zhou, 1986). Nowadays, there are tremendous interests to apply DTM in analysis of uniform and non-uniform beams (Ozdemir & Kaya, 2006; Ozgumus & Kaya, 2006; Seval & Çatal, 2008; Mei, 2008). Recent study also shows that the DTM can be applied in free vibration analysis of rotating non-prismatic beams (Attarnejad & Shahba, 2008).

In this paper, we generalized the differential transformation of the boundary value problem with linear differential equation to n-th order. By implementing this proposed generalization, we show that the polynomial function can be solved easily, efficiently and accurately.

DIFFERENTIAL TRANSFORMATION METHOD

Suppose that, the function $f(x)$ is a continuously differentiable function on the interval $(x_0 - r, x_0 + r)$.

Definition 2

The differential transform of the function $f(x)$ for the k-th derivative is defined as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{3}$$

where $f(x)$ is the original function and $F(k)$ is the transformed function.

Definition 3

The inverse differential transform of $F(k)$ is defined as follows:

$$f(x) = \sum_{k=0}^{\infty} (x-x_0)^k F(k) \tag{4}$$

Substitution of equation (3) into equation (4) yields:

$$f(x) = \sum_{k=0}^{\infty}(x-x_0)^k \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{5}$$

Note that, this is the Taylor series of $f(x)$ at $x = x_0$. The following basic operations of differential transformation can be deduced from equations (3) and (4):

a) If $t(x) = r(x) \pm p(x)$ then, $T(k) = R(k) \pm P(k)$.
b) If $t(x) = \alpha r(x)$ then, $T(k) = \alpha R(k)$.
c) If $t(x) = \frac{dr(x)}{dx}$ then, $T(k) = (k+1)R(k+1)$.
d) If $t(x) = \frac{d^2 r(x)}{dx^2}$ then, $T(k) = (k+1)(k+2)R(k+2)$.
e) If $t(x) = \frac{d^b r(x)}{dx^b}$ then, $T(k) = (k+1)(k+2)\cdots(k+b)R(k+b)$.
f) If $t(x) = r(x)p(x)$ then, $T(k) = \sum_{l=0}^{k} P(l)R(k-l)$.
g) If $t(x) = x^b$ then $T(k) = \delta(k-b)$ where, $\delta(k-b) = \begin{cases} 1, & if\ k = b \\ 0, & if\ k \neq b \end{cases}$
h) If $t(x) = \exp(\lambda x)$ then, $T(k) = \frac{\lambda^k}{k!}$
i) If $t(x) = (1+x)^b$ then, $T(k) = \frac{b(b-1)\cdots(b-k+1)}{k!}$
j) If $t(x) = sin(jx+\alpha)$ then, $T(k) = \frac{j^k}{k!} sin\left(\frac{\pi k}{2} + \alpha\right)$.
k) If $t(x) = cos(jx+\alpha)$ then, $T(k) = \frac{j^k}{k!} cos\left(\frac{\pi k}{2} + \alpha\right)$.

More details on the DTM are available in (Erturk & Momani, 2007).

PROPOSED METHOD

We prove the following theorem by using induction method.

Theorem 2

The general differential transformation for BVP of linear differential equation, $f^{(n)}(x) = f(x) + [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_m x^m]e^x$ *is given by*

$$F(n+k) = \frac{k!}{(n+k)!}\left[F(k) + \frac{a_0}{k!} + \sum_{i=1}^{m}\left[a_i\left(\sum_{l=0}^{k}\frac{\delta(l-i)}{(k-l)!}\right)\right]\right] \tag{6}$$

for integer $n \geq 1$.

Proof:

For $m = 1$, *we can derive*

$$f^{(n)}(x) = f(x) + a_0 e^x + a_1 x e^x \tag{7}$$

The derivative of equation (7) for $k \in \mathbb{Z}$ *is*

$$f^{(n+k)}(x) = f^{(k)}(x) + (a_0 + k a_1)e^x + a_1 x e^x$$

Then,

$$f^{(n+k)}(x)|_{x=0} = f^{(k)}(x) + a_0 + ka_1$$

By Definition 2, we have

$$(n+k)!\,F(n+k) = k!\,F(k) + a_0 + ka_1$$

$$F(n+k) = \frac{k!\left[F(k) + \frac{a_0}{k!} + \frac{a_1}{(k-1)!}\right]}{(n+k)!} \tag{8}$$

For $m = 1$, equation (6) gives

$$F(n+k) = \frac{k!}{(n+k)!}\left[F(k) + \frac{a_0}{k!} + \sum_{i=1}^{1}\left[a_i\left(\sum_{l=0}^{k}\frac{\delta(l-i)}{(k-l)!}\right)\right]\right]$$

$$F(n+k) = \frac{k!}{(n+k)!}\left[F(k) + \frac{a_0}{k!} + \sum_{i=1}^{1}\left[a_i\left(\ldots + \frac{1}{(k-1)!} + \cdots\right)\right]\right]$$

$$F(n+k) = \frac{k!\left[F(k) + \frac{a_0}{k!} + \frac{a_1}{(k-1)!}\right]}{(n+k)!} \tag{9}$$

Since equation (8) *is equal to equation* (9)*, Theorem 2 holds for* $m = 1$.

Assume that Theorem 2 holds for $m = p$. That is, the differential transformation of

$$f^{(n)}(x) = f(x) + a_0e^x + a_1xe^x + a_2x^2e^x + a_3x^3e^x + \cdots + a_px^pe^x \tag{10}$$

is given by

$$F(n+k) = \frac{k!}{(n+k)!}\left[F(k) + \frac{a_0}{k!} + \sum_{i=1}^{p}\left[a_i\left(\sum_{l=0}^{k}\frac{\delta(l-i)}{(k-l)!}\right)\right]\right] \tag{11}$$

Then for $m = p + 1$, we have

$$f^{(n)}(x) = f(x) + \left[a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_px^p + a_{p+1}x^{p+1}\right]e^x \tag{12}$$

The derivative of equation (12) for $k \in \mathbb{Z}$ is

$$\begin{aligned} f^{(n+k)}(x)|_{x=0} &= f^{(k)}(x) + a_0 + ka_1 + k(k-1)a_2 + k(k-1)(k-2)a_3 + \cdots \\ &\quad + k(k-1)(k-2)\cdots(k-p-1)a_p + k(k-1)(k-2)\cdots(k-p)a_{p+1} \end{aligned}$$

By Definition 2, we have

$$
\begin{aligned}
&(n+k)!\,F(n+k) \\
&\qquad = k!\,F(k) + a_0 + ka_1 + k(k-1)a_2 + k(k-1)(k-2)a_3 + \cdots \\
&\qquad + k(k-1)(k-2)\cdots(k-p-1)a_p + k(k-1)(k-2)\cdots(k-p)a_{p+1}
\end{aligned}
$$

$$
F(n+k) = \frac{k!\left[F(k) + \frac{a_0}{k!} + \frac{k}{k!}a_1 + \frac{k(k-1)}{k!}a_2 + \cdots + \frac{k(k-1)(k-2)\cdots(k-p)}{k!}a_{p+1}\right]}{(n+k)!}
$$

$$
F(n+k) = \frac{k!\left[F(k) + \frac{a_0}{k!} + \frac{a_1}{(k-1)!} + \cdots + \frac{a_p}{(k-p)!} + + \frac{a_{p+1}}{(k-p-1)!}\right]}{(n+k)!} \tag{13}
$$

For $m = p + 1$, *equation (6) gives*

$$
F(n+k) = \frac{k!}{(n+k)!}\left[F(k) + \frac{a_0}{k!} + \sum_{i=1}^{p+1}\left[a_i\left(\sum_{l=0}^{k}\frac{\delta(l-i)}{(k-l)!}\right)\right]\right]
$$

From equation (11), we have

$$
F(n+k) = \frac{k!}{(n+k)!}\left[F(k) + \frac{a_0}{k!} + \sum_{i=1}^{p}\left[a_i\left(\sum_{l=0}^{k}\frac{\delta(l-i)}{(k-l)!}\right)\right] + a_{p+1}\left(\sum_{l=0}^{k}\frac{\delta(l-i)}{(k-l)!}\right)\right]
$$

$$
F(n+k) = \frac{k!}{(n+k)!}\left[F(k) + \frac{a_0}{k!} + \frac{a_1}{(k-1)!} + \cdots + \frac{a_p}{(k-p)!} + \frac{a_{p+1}}{(k-p-1)!}\right] \tag{14}
$$

Note that, we have equation (13) *is equal to equation* (14). *This implies that, Theorem 2 holds for* $m = p + 1$.

Now, we prove the general form of n*-th order BVPs. For that purpose, we fix* $m = 1$.

For $n = 1$, *we have*

$$
f^{(')}(x) = f(x) + a_0 e^x + a_1 x e^x \tag{15}
$$

The derivative of equation (15) for $k \in \mathbb{Z}$ *is*

$$
f^{(1+k)}(x) = f^{(k)}(x) + (a_0 + ka_1)e^x + a_1 x e^x
$$

Then,

$$
f^{(1+k)}(x)|_{x=0} = f^{(k)}(x) + a_0 + ka_1
$$

By Definition 2, we have

$$
(1+k)!\,F(1+k) = k!\,F(k) + a_0 + ka_1
$$

$$F(1+k)=\frac{k!\left[F(k)+\frac{a_0}{k!}+\frac{a_1}{(k-1)!}\right]}{(1+k)!} \tag{16}$$

For $n=1$, equation (6) gives

$$F(1+k)=\frac{k!}{(1+k)!}\left[F(k)+\frac{a_0}{k!}+a_1\left(\sum_{l=0}^{k}\frac{\delta(l-i)}{(k-l)!}\right)\right]$$

$$F(1+k)=\frac{k!\left[F(k)+\frac{a_0}{k!}+\frac{a_1}{(k-1)!}\right]}{(1+k)!} \tag{17}$$

Note that, we have equation (16) *is equal to equation* (17). *Hence, Theorem 2 holds for* $n=1$.

Assume that, Theorem 2 holds for $n=q$. *Thus, for* $n=q+1$ *we have*

$$f^{(q+1+k)}(x)|_{x=0}=f^{(k)}(x)+a_0+ka_1 \tag{18}$$

By Definition 2, we have

$$(q+1+k)!\,F(q+1+k)=k!\,F(k)+a_0+ka_1$$

$$F(q+1+k)=\frac{k!}{(q+1+k)!}\left(F(k)+\frac{a_0}{k!}+\frac{a_1}{(k-1)!}\right) \tag{19}$$

For $n=q+1$, equation (6) gives

$$F(q+1+k)=\frac{k!}{(q+1+k)!}\left[F(k)+\frac{a_0}{k!}+a_1\left(\sum_{l=0}^{k}\frac{\delta(l-i)}{(k-l)!}\right)\right]$$

$$F(q+1+k)=\frac{k!\left[F(k)+\frac{a_0}{k!}+\frac{a_1}{(k-1)!}\right]}{(q+1+k)!} \tag{20}$$

Note that, we have equation (19) *is equal to equation* (20). *This implies that, Theorem 2 holds for* $n=q+1$. ■

NUMERICAL EXAMPLES

Consider the following numerical examples:

Example 1
Consider the following fifth-order boundary value problems of second-order linear differential equation,

$$f^{(5)}(x)=f(x)-10e^x-5xe^x+x^2e^x \tag{21}$$

subject to the boundary conditions,

$$f(0) = 0, \quad f'(0) = \frac{4}{5}, \quad f''(0) = \frac{1}{5}$$

$$f'(1) = -2.265234857, \quad f''(1) = -8.245454881 \tag{22}$$

By Theorem 2, the equation (21) is transformed to equation (23),

$$F(k+5) = \frac{k!}{(k+5)!}\left[F(k) - \frac{10}{k!} - 5\left(\sum_{l=0}^{k} \frac{\delta(l-1)}{(k-l)!}\right) + \left(\sum_{l=0}^{k} \frac{\delta(l-2)}{(k-l)!}\right)\right] \tag{23}$$

Applying Definition 2 on the boundary value problem (22) at $x = 0$, the following transformed boundary conditions can be obtained,

$$F(0) = 0, \quad F(1) = \frac{4}{5}, \quad F(2) = -\frac{1}{10} \tag{24}$$

where $a = \frac{f'''(x)}{3!} = F(3)$ and $b = \frac{f^{(4)}(x)}{4!} = F(4)$.

By utilizing the transformed equation (23) and boundary conditions (24), we can easily get the solution of $F(r)$, for $r \geq 5$. We can get the constants a and b by using the boundary conditions (22) at $x = 1$ and solve the following system,

$$\frac{15443560997}{435891456000} + \frac{239595841}{79833600}a + \frac{148284463}{37065600}b = -2.265234857 \tag{25}$$

$$-\frac{20554253119}{7783776000} + \frac{39972241}{6652800}a + \frac{239595841}{19958400}b = -8.245454881 \tag{26}$$

This system gives $a = -0.4333333329$ and $b = -0.2500000003$.

As a result, the following series solution can be formed by applying the inverse transformation in Definition 3 up to $N = 15$.

$$\begin{aligned} f(x) = {} & 0.8x - 0.1x^2 - 0.4333333329x^3 - 0.2500000003x^4 - 0.08333333333x^5 - \\ & 0.01972222222x^6 - 0.003611111111x^7 - 0.0005357142856x^8 - \\ & 0.00006613756615x^9 - 0.000006889329806x^{10} - 6.062610229 \times 10^{-7}x^{11} - \\ & 4.425872481 \times 10^{-8}x^{12} - 2.505210838 \times 10^{-9}x^{13} - 8.029521927 \times 10^{-11}x^{14} + \\ & 3.823581866 \times 10^{-12}x^{15} \end{aligned}$$

The results of the Example 1 together with the exact solutions are presented in Table 1:

Table 1. Numerical result for Example 1

x	*Exact solution*	*(DTM N=15)*	*Error*
0.1	0.1521053568	0.1521053568	0
0.2	0.2170572964	0.2170572964	0

0.3	0.2170572964	0.2170572964	0
0.4	0.2689262657	0.2689262657	0
0.5	0.3022655663	0.3022655664	1×10^{-10}
0.6	0.3104890436	0.3104890437	1×10^{-10}
0.7	0.2856843842	0.2856843842	0
0.8	0.2183997500	0.2183997501	1×10^{-10}
0.9	0.09740028320	0.09740028330	1.0×10^{10}
1.0	-0.09060939426	-0.09060939416	1.0×10^{10}

Example 2
Consider the following sixth-order boundary value problem of third-order linear differential equation,

$$f^{(6)}(x) = f(x) + 15e^x + 10xe^x + x^3e^x \tag{27}$$

subject to the boundary conditions,

$$f(0) = 0, \quad f^{'}(0) = -\frac{25}{8}, \quad f^{''}(0) = -\frac{5}{3}$$

$$f(1) = -3.397852287, \quad f^{'}(1) = -2.831543570, \quad f^{''}(0) = 3.397852297 \tag{28}$$

By Theorem 2, the equation (27) is transformed to the following equation,

$$F(k+6) = \frac{k!}{(k+6)!}\left[F(k) + \frac{15}{k!} + 10\left(\sum_{l=0}^{k}\frac{\delta(l-1)}{(k-l)!}\right) + \left(\sum_{l=0}^{k}\frac{\delta(l-3)}{(k-l)!}\right)\right] \tag{29}$$

Applying Definition 2 on the boundary value problem (28) at $x = 0$, the following transformed boundary conditions can be obtained,

$$F(0) = 0, \quad F(1) = -\frac{25}{8}, \quad F(2) = -\frac{5}{6} \tag{30}$$

where $= \frac{f^{'''}(x)}{3!} = F(3)$, $b = \frac{f^{(4)}(x)}{4!} = F(4)$ and $c = \frac{f^{(5)}(x)}{5!} = F(5)$.
By utilizing the transformed equation (29) and transformed boundary conditions (30), we can easily get the solution of $F(r)$, for $r \geq 6$. We can get the constants a, b and c by using the boundary conditions (28) at $x = 1$ and solve the following system,

$$-\frac{4113595263593}{1046139494400} + \frac{217949331601}{217945728000}a + \frac{151201}{151200}b + \frac{332641}{332640}c = -3.397852287 \tag{31}$$

$$-\frac{4841688336097}{1046139494400} + \frac{43591307761}{14529715200}a + \frac{60481}{15120}b + \frac{151201}{30240}c = -2.83154357 \tag{32}$$

$$-\frac{29911705387}{37362124800} + \frac{889750903}{148262400}a + \frac{20161}{1680}b + \frac{60481}{3024}c = 3.397852297 \tag{33}$$

This system gives $a = 0.3124999935,\quad b = 0.2500000111$ and $c = 0.08506943967.$
As a result, the following series solution can be formed by applying the inverse transformation Definition 3 up to $N = 15.$

$$f(x) = -3.125x - 0.8333333333x^2 + 0.3124999935x^3 + 0.2500000111x^4 + 0.08506943967x^5 + 0.02083333333x^6 + 0.004340277778x^7 + 0.0008267195767x^8 + 0.0001457093252x^9 + 0.00002342372142x^{10} + 0.000003387253807x^{11} + 4.384118967 \times 10^{-7}x^{12} + 5.088709516 \times 10^{-8}x^{13} + 5.326249539 \times 10^{-9}x^{14} + 5.061466495 \times 10^{-10}x^{15}$$

The results of the Example 2 together with the exact solutions are presented in Table 2:

Table 2. Numerical result for Example 2

x	***Exact solution***	***(DTM N=15)***	***Error***
0.1	-0.3204949614	-0.3204949614	0
0.2	-0.6554047200	-0.6554047201	1×10^{-10}
0.3	-1.001814588	-1.001814588	0
0.4	-1.355969196	-1.355969196	0
0.5	-1.713124445	-1.713124445	0
0.6	-2.067375991	-2.067375991	0
0.7	-2.411460475	-2.411460475	0
0.8	-2.736525125	-2.736525126	1×10^{-9}
0.9	-3.031860520	-3.031860522	2×10^{-9}
1.0	-3.284590543	-3.284590543	0

Example 3

Consider the following fifth-order boundary value problem of fourth-order linear differential equation,

$$f^{(5)}(x) = f(x) - 15e^x + 10x^4e^x \tag{34}$$

subject to the boundary conditions,

$$f(0) = 0, \quad f'(0) = -\frac{63}{5}, \quad f''(0) = -\frac{366}{5}$$
$$f(1) = -65.78242029, \quad f'(1) = -138.0887175 \tag{35}$$

By Theorem 2, the equation (34) is transformed to the following equation,

$$F(k+5) = \frac{k!}{(k+5)!}\left[F(k) - \frac{15}{k!} + 10\left(\sum_{l=0}^{k} \frac{\delta(l-4)}{(k-l)!}\right)\right] \tag{36}$$

Applying Definition 2 on the boundary value problem (35) at $x = 0$, the following transformed boundary conditions can be obtained,

$$F(0) = 0, \qquad F(1) = -\frac{63}{5}, \qquad F(2) = -\frac{183}{5} \tag{37}$$

where $a = \frac{f'''(x)}{3!} = F(3)$ and $b = \frac{f^{(4)}(x)}{4!} = F(4)$.
By utilizing the transformed equation (36) and transformed boundary conditions (37), we can easily get the solution of $F(r)$, for $r \geq 5$. We can get the constants a, b and c by using the boundary conditions (35) at $x = 1$ and solve the following system,

$$-\frac{279537492877}{5660928000} + \frac{148284463}{148262400}a + \frac{3632669041}{3632428800}b = -65.78242029 \tag{38}$$

$$-\frac{257295882443}{2965248000} + \frac{239595841}{79833600}a + \frac{148284463}{37065600}b = -138.0887175 \tag{39}$$

This system gives $a = -14.29999960$ and $b = -2.100000434$.

As a result, the following series solution can be formed by applying the inverse transformation Definition 3 up to $N = 15$.

$$f(x) = -12.6x - 36.6x^2 - 14.2999996x^3 - 2.100000434x^4 - 0.125x^5 - 0.03833333333x^6 - 0.0175x^7 - 0.00249999994x^8 + 0.0004811507649x^9 + 0.0003224206349x^{10} + 0.00008912037037x^{11} + 0.00001732102774x^{12} + 0.000002679322992x^{13} + 3.487060777 \times 10^{-7}x^{14} + 3.942495262 \times 10^{-8}x^{15}$$

The results of the Example 3 together with the exact solutions are presented in Table 3 below:

Table 3. Numerical result for Example 3

x	***Exact solution***	***(DTM N=15)***	***Error***
0.1	-1.640511290	-1.640511290	0
0.2	-4.101802683	-4.101802681	2×10^{-9}
0.3	-7.477445675	-7.477445668	7×10^{-9}
0.4	-11.86642716	-11.86642714	2×10^{-8}
0.5	-17.37340041	-17.37340039	2×10^{-8}
0.6	-24.10899320	-24.10899317	3×10^{-8}
0.7	-32.19018339	-32.19018335	4×10^{-8}
0.8	-41.74075008	-41.74075006	2×10^{-8}
0.9	-52.89180708	-52.89180707	1×10^{-8}
1.0	-65.78242029	-65.78242028	1×10^{-8}

CONCLUSION

We extended and proved the theorem for n-th order boundary value problems of m-th order linear differential equation. We also provided several numerical examples to clarify our theorem. From the results, we observe that the proposed method, as well as the differential transformation method have good accuracy and fast convergent. In addition, the method is efficient and requires less computational cost for solving numerical solution in the bounded domains. Therefore, we conclude that, the proposed method is a successful numerical method.

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